A Hierarchy of Algebras for Boolean Subsets

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Abstract

We present a collection of axiom systems for the construction of Boolean subalgebras of larger overall algebras. The subalgebras are defined as the range of a complement-like operation on a semilattice. This technique has been used, for example, with the antidomain operation, dynamic negation and Stone algebras. We present a common ground for these constructions based on a new equational axiomatisation of Boolean algebras.

Contents

1 Overview

A Boolean algebra often arises as a subalgebra of some overall algebra. To avoid introducing a separate type for the subalgebra, the overall algebra can be enriched with a special operation leading into the intended subalgebra and axioms to guarantee that the range of this operation has a Boolean structure. Examples for this are the antidomain operation in idempotent (left) semirings [6, 7, 8], dynamic negation [17], the operation yielding tests in [13, 16], and the pseudocomplement operation in Stone algebras [9, 12, 14]. The present development looks at a common ground pattern.

In Sections 2 and 3 we relate various axiomatisations of Boolean algebras from the literature and present a new equational one tailored to our needs. Section 4 adapts this for the construction of Boolean subalgebras of larger overall algebras. In Section 5 we add successively stronger assumptions to the overall algebra. Sections 6, 7 and 8 show how Stone algebras, domain semirings and antidomain semirings fit into this hierarchy.

This Isabelle/HOL theory formally verifies results in [15]. See that paper for further details and related work. Some proofs in this theory have been translated from proofs found by Prover9 [21] using a program we wrote.

theory *Subset-Boolean-Algebras*

imports *Stone-Algebras.P-Algebras*

begin

2 Boolean Algebras

We show that Isabelle/HOL's *boolean-algebra* class is equivalent to Huntington's axioms [18]. See [24] for related results.

2.1 Huntington's Axioms

Definition 1

```
class huntington = sup + uminus +
  assumes associative: x ⊔ (y ⊔ z) = (x ⊔ y) ⊔ z
  assumes commutative: x ⊔ y = y ⊔ x
  assumes huntington: x = −(−x ⊔ y) ⊔ −(−x ⊔ −y)
begin

lemma top-unique:
  x ⊔ −x = y ⊔ −y
proof −
  have x ⊔ −x = y ⊔ −(−−y ⊔ −x) ⊔ −(−−y ⊔ −−x)
    by (smt associative commutative huntington)
  thus ?thesis
    by (metis associative huntington)
qed

end
```

2.2 Equivalence to *boolean-algebra* Class

Definition 2

```
class extended = sup + inf + minus + uminus + bot + top + ord +
  assumes top-def: top = (THE x . ∀ y . x = y ⊔ −y)
  assumes bot-def: bot = −(THE x . ∀ y . x = y ⊔ −y)
  assumes inf-def: x ⊓ y = −(−x ⊔ −y)
  assumes minus-def: x − y = −(−x ⊔ y)
  assumes less-eq-def: x ≤ y ⟷ x ⊔ y = y
  assumes less-def: x < y ⟷ x ⊔ y = y ∧ ¬ (y ⊔ x = x)

class huntington-extended = huntington + extended
begin

lemma top-char:
  top = x ⊔ −x
  using top-def top-unique by auto

lemma bot-char:
  bot = −top
  by (simp add: bot-def top-def)

subclass boolean-algebra
proof
  show 1: ⋀x y. (x < y) = (x ≤ y ∧ ¬ y ≤ x)
    by (simp add: less-def less-eq-def)
  show 2: ⋀x. x ≤ x
  proof −
    fix x
```

have $x \sqcup top = top \sqcup --x$
by (*metis* (*full-types*) *associative top-char*)
thus $x \leq x$
by (*metis* (*no-types*) *associative huntington less-eq-def top-char*)
qed
show *3*: $\bigwedge x\ y\ z.\ x \leq y \Longrightarrow y \leq z \Longrightarrow x \leq z$
by (*metis associative less-eq-def*)
show *4*: $\bigwedge x\ y.\ x \leq y \Longrightarrow y \leq x \Longrightarrow x = y$
by (*simp add*: *commutative less-eq-def*)
show *5*: $\bigwedge x\ y.\ x \sqcap y \leq x$
using *2* **by** (*metis associative huntington inf-def less-eq-def*)
show *6*: $\bigwedge x\ y.\ x \sqcap y \leq y$
using *5 commutative inf-def* **by** *fastforce*
show *8*: $\bigwedge x\ y.\ x \leq x \sqcup y$
using *2 associative less-eq-def* **by** *auto*
show *9*: $\bigwedge y\ x.\ y \leq x \sqcup y$
using *8 commutative* **by** *fastforce*
show *10*: $\bigwedge y\ x\ z.\ y \leq x \Longrightarrow z \leq x \Longrightarrow y \sqcup z \leq x$
by (*metis associative less-eq-def*)
show *11*: $\bigwedge x.\ bot \leq x$
using *8* **by** (*metis bot-char huntington top-char*)
show *12*: $\bigwedge x.\ x \leq top$
using *6 11* **by** (*metis huntington bot-def inf-def less-eq-def top-def*)
show *13*: $\bigwedge x\ y\ z.\ x \sqcup y \sqcap z = (x \sqcup y) \sqcap (x \sqcup z)$
proof −
have *2*: $\bigwedge x\ y\ z\ .\ x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$
by (*simp add*: *associative*)
have *3*: $\bigwedge x\ y\ z\ .\ (x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$
using *2* **by** *metis*
have *4*: $\bigwedge x\ y\ .\ x \sqcup y = y \sqcup x$
by (*simp add*: *commutative*)
have *5*: $\bigwedge x\ y\ .\ x = -\ (-\ x \sqcup y) \sqcup -\ (-\ x \sqcup -\ y)$
by (*simp add*: *huntington*)
have *6*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcup y) \sqcup -\ (-\ x \sqcup -\ y) = x$
using *5* **by** *metis*
have *7*: $\bigwedge x\ y\ .\ x \sqcap y = -\ (-\ x \sqcup -\ y)$
by (*simp add*: *inf-def*)
have *10*: $\bigwedge x\ y\ z\ .\ x \sqcup (y \sqcup z) = y \sqcup (x \sqcup z)$
using *3 4* **by** *metis*
have *11*: $\bigwedge x\ y\ z\ .\ -\ (-\ x \sqcup y) \sqcup (-\ (-\ x \sqcup -\ y) \sqcup z) = x \sqcup z$
using *3 6* **by** *metis*
have *12*: $\bigwedge x\ y\ .\ -\ (x \sqcup -\ y) \sqcup -\ (-\ y \sqcup -\ x) = y$
using *4 6* **by** *metis*
have *13*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcup y) \sqcup -\ (-\ y \sqcup -\ x) = x$
using *4 6* **by** *metis*
have *14*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ (-\ (-\ x \sqcup y) \sqcup -\ -\ (-\ x \sqcup -\ y)) = -\ x \sqcup y$
using *6* **by** *metis*
have *18*: $\bigwedge x\ y\ z\ .\ -\ (x \sqcup -\ y) \sqcup (-\ (-\ y \sqcup -\ x) \sqcup z) = y \sqcup z$
using *3 12* **by** *metis*

have *20*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcup -\ y) \sqcup -\ (y \sqcup -\ x) = x$
using *4 12* **by** *metis*
have *21*: $\bigwedge x\ y\ .\ -\ (x \sqcup -\ y) \sqcup -\ (-\ x \sqcup -\ y) = y$
using *4 12* **by** *metis*
have *22*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ (-\ (y \sqcup -\ x) \sqcup -\ -\ (-\ x \sqcup -\ y)) = y \sqcup -\ x$
using *6 12* **by** *metis*
have *23*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ (-\ x \sqcup (-\ y \sqcup -\ (y \sqcup -\ x))) = y \sqcup -\ x$
using *3 4 6 12* **by** *metis*
have *24*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ (-\ (-\ x \sqcup -\ y) \sqcup -\ -\ (-\ x \sqcup y)) = -\ x \sqcup -\ y$
using *6 12* **by** *metis*
have *28*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcup -\ y) \sqcup -\ (-\ y \sqcup x) = y$
using *4 13* **by** *metis*
have *30*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ (-\ y \sqcup (-\ x \sqcup -\ (-\ x \sqcup y))) = -\ x \sqcup y$
using *3 4 6 13* **by** *metis*
have *32*: $\bigwedge x\ y\ z\ .\ -\ (-\ x \sqcup y) \sqcup (z \sqcup -\ (-\ y \sqcup -\ x)) = z \sqcup x$
using *10 13* **by** *metis*
have *37*: $\bigwedge x\ y\ z\ .\ -\ (-\ x \sqcup -\ y) \sqcup (-\ (y \sqcup -\ x) \sqcup z) = x \sqcup z$
using *3 20* **by** *metis*
have *39*: $\bigwedge x\ y\ z\ .\ -\ (-\ x \sqcup -\ y) \sqcup (z \sqcup -\ (y \sqcup -\ x)) = z \sqcup x$
using *10 20* **by** *metis*
have *40*: $\bigwedge x\ y\ z\ .\ -\ (x \sqcup -\ y) \sqcup (-\ (-\ x \sqcup -\ y) \sqcup z) = y \sqcup z$
using *3 21* **by** *metis*
have *43*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ (-\ y \sqcup (-\ x \sqcup -\ (y \sqcup -\ x))) = y \sqcup -\ x$
using *3 4 6 21* **by** *metis*
have *47*: $\bigwedge x\ y\ z\ .\ -\ (x \sqcup y) \sqcup -\ (-\ (-\ x \sqcup z) \sqcup -\ (-\ (-\ x \sqcup -\ z) \sqcup y)) =$
$-\ x \sqcup z$
using *6 11* **by** *metis*
have *55*: $\bigwedge x\ y\ .\ x \sqcup -\ (-\ y \sqcup -\ -\ x) = y \sqcup -\ (-\ x \sqcup y)$
using *4 11 12* **by** *metis*
have *58*: $\bigwedge x\ y\ .\ x \sqcup -\ (-\ -\ y \sqcup -\ x) = x \sqcup -\ (-\ x \sqcup y)$
using *4 11 13* **by** *metis*
have *63*: $\bigwedge x\ y\ .\ x \sqcup -\ (-\ -\ x \sqcup -\ y) = y \sqcup -\ (-\ x \sqcup y)$
using *4 11 21* **by** *metis*
have *71*: $\bigwedge x\ y\ .\ x \sqcup -\ (-\ y \sqcup x) = y \sqcup -\ (-\ x \sqcup y)$
using *4 11 28* **by** *metis*
have *75*: $\bigwedge x\ y\ .\ x \sqcup -\ (-\ y \sqcup x) = y \sqcup -\ (y \sqcup -\ x)$
using *4 71* **by** *metis*
have *78*: $\bigwedge x\ y\ .\ -\ x \sqcup (y \sqcup -\ (-\ x \sqcup (y \sqcup -\ -\ (-\ x \sqcup -\ y)))) = -\ x \sqcup -$
$(-\ x \sqcup -\ y)$
using *3 4 6 71* **by** *metis*
have *86*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcup -\ (-\ y \sqcup x)) \sqcup -\ (y \sqcup -\ (-\ x \sqcup y)) = -\ y \sqcup x$
using *4 20 71* **by** *metis*
have *172*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ (-\ x \sqcup -\ y) = y \sqcup -\ (-\ -\ x \sqcup y)$
using *14 75* **by** *metis*
have *201*: $\bigwedge x\ y\ .\ x \sqcup -\ (-\ y \sqcup -\ -\ x) = y \sqcup -\ (y \sqcup -\ x)$
using *4 55* **by** *metis*
have *236*: $\bigwedge x\ y\ .\ x \sqcup -\ (-\ -\ y \sqcup -\ x) = x \sqcup -\ (y \sqcup -\ x)$
using *4 58* **by** *metis*
have *266*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ (-\ (-\ x \sqcup -\ (y \sqcup -\ -\ x)) \sqcup -\ -\ (-\ x \sqcup -\ -\ (-$

$- x \sqcup y))) = - x \sqcup - (- - x \sqcup y)$
using *14 58 236* **by** *metis*
have *678*: $\bigwedge x\ y\ z\ .\ - (- x \sqcup - (- y \sqcup x)) \sqcup (- (y \sqcup - (- x \sqcup y)) \sqcup z) = - y \sqcup (x \sqcup z)$
using *3 4 37 71* **by** *smt*
have *745*: $\bigwedge x\ y\ z\ .\ - (- x \sqcup - (- y \sqcup x)) \sqcup (z \sqcup - (y \sqcup - (- x \sqcup y))) = z \sqcup (- y \sqcup x)$
using *4 39 71* **by** *metis*
have *800*: $\bigwedge x\ y\ .\ - - x \sqcup (- y \sqcup (- (y \sqcup - - x) \sqcup - (- x \sqcup (- - x \sqcup (- y \sqcup - (y \sqcup - - x)))))) = x \sqcup - (y \sqcup - - x)$
using *3 23 63* **by** *metis*
have *944*: $\bigwedge x\ y\ .\ x \sqcup - (x \sqcup - - (- (- x \sqcup - y) \sqcup - - (- x \sqcup y))) = - (- x \sqcup - y) \sqcup - (- (- x \sqcup - y) \sqcup - - (- x \sqcup y))$
using *4 24 71* **by** *metis*
have *948*: $\bigwedge x\ y\ .\ - x \sqcup - (- (y \sqcup - (y \sqcup - - x)) \sqcup - - (- x \sqcup (- y \sqcup - x))) = - x \sqcup - (- y \sqcup - x)$
using *24 75* **by** *metis*
have *950*: $\bigwedge x\ y\ .\ - x \sqcup - (- (y \sqcup - (- - x \sqcup y)) \sqcup - - (- x \sqcup (- x \sqcup - y))) = - x \sqcup - (- x \sqcup - y)$
using *24 75* **by** *metis*
have *961*: $\bigwedge x\ y\ .\ - x \sqcup - (- (y \sqcup - (- - x \sqcup y)) \sqcup - - (- x \sqcup (- - - x \sqcup - y))) = y \sqcup - (- - x \sqcup y)$
using *24 63* **by** *metis*
have *966*: $\bigwedge x\ y\ .\ - x \sqcup - (- (y \sqcup - (y \sqcup - - x)) \sqcup - - (- x \sqcup (- y \sqcup - - - x))) = y \sqcup - (y \sqcup - - x)$
using *24 201* **by** *metis*
have *969*: $\bigwedge x\ y\ .\ - x \sqcup - (- (- x \sqcup - (y \sqcup - - x)) \sqcup - - (- x \sqcup (- - y \sqcup - - x))) = - x \sqcup - (y \sqcup - - x)$
using *24 236* **by** *metis*
have *1096*: $\bigwedge x\ y\ z\ .\ - x \sqcup (- (- x \sqcup - y) \sqcup z) = y \sqcup (- (- - x \sqcup y) \sqcup z)$
using *3 172* **by** *metis*
have *1098*: $\bigwedge x\ y\ z\ .\ - x \sqcup (y \sqcup - (- x \sqcup - z)) = y \sqcup (z \sqcup - (- - x \sqcup z))$
using *10 172* **by** *metis*
have *1105*: $\bigwedge x\ y\ .\ x \sqcup - x = y \sqcup - y$
using *4 10 12 32 172* **by** *metis*
have *1109*: $\bigwedge x\ y\ z\ .\ x \sqcup (- x \sqcup y) = z \sqcup (- z \sqcup y)$
using *3 1105* **by** *metis*
have *1110*: $\bigwedge x\ y\ z\ .\ x \sqcup - x = y \sqcup (z \sqcup - (y \sqcup z))$
using *3 1105* **by** *metis*
have *1114*: $\bigwedge x\ y\ .\ - (- x \sqcup - - x) = - (y \sqcup - y)$
using *7 1105* **by** *metis*
have *1115*: $\bigwedge x\ y\ z\ .\ x \sqcup (y \sqcup - y) = z \sqcup (x \sqcup - z)$
using *10 1105* **by** *metis*
have *1117*: $\bigwedge x\ y\ .\ - (x \sqcup - - x) \sqcup - (y \sqcup - y) = - x$
using *4 13 1105* **by** *metis*
have *1121*: $\bigwedge x\ y\ .\ - (x \sqcup - x) \sqcup - (y \sqcup - - y) = - y$
using *4 28 1105* **by** *metis*
have *1122*: $\bigwedge x\ .\ - - x = x$
using *4 28 1105 1117* **by** *metis*

have *1134*: $\bigwedge x\ y\ z\ .\ -\ (x \sqcup -\ y) \sqcup (z \sqcup -\ z) = y \sqcup (-\ y \sqcup -\ x)$
using *18 1105 1122* **by** *metis*
have *1140*: $\bigwedge x\ .\ -\ x \sqcup -\ (x \sqcup (x \sqcup -\ x)) = -\ x \sqcup -\ x$
using *4 22 1105 1122 1134* **by** *metis*
have *1143*: $\bigwedge x\ y\ .\ x \sqcup (-\ x \sqcup y) = y \sqcup (x \sqcup -\ y)$
using *37 1105 1122 1134* **by** *metis*
have *1155*: $\bigwedge x\ y\ .\ -\ (x \sqcup -\ x) \sqcup -\ (y \sqcup y) = -\ y$
using *1121 1122* **by** *metis*
have *1156*: $\bigwedge x\ y\ .\ -\ (x \sqcup x) \sqcup -\ (y \sqcup -\ y) = -\ x$
using *1117 1122* **by** *metis*
have *1157*: $\bigwedge x\ y\ .\ -\ (x \sqcup -\ x) = -\ (y \sqcup -\ y)$
using *4 1114 1122* **by** *metis*
have *1167*: $\bigwedge x\ y\ z\ .\ -\ x \sqcup (y \sqcup -\ (-\ x \sqcup -\ z)) = y \sqcup (z \sqcup -\ (x \sqcup z))$
using *1098 1122* **by** *metis*
have *1169*: $\bigwedge x\ y\ z\ .\ -\ x \sqcup (-\ (-\ x \sqcup -\ y) \sqcup z) = y \sqcup (-\ (x \sqcup y) \sqcup z)$
using *1096 1122* **by** *metis*
have *1227*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ (-\ x \sqcup (y \sqcup (x \sqcup -\ (-\ x \sqcup -\ (y \sqcup x))))) = -\ x \sqcup -\ (y \sqcup x)$
using *3 4 969 1122* **by** *smt*
have *1230*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ (-\ x \sqcup (-\ y \sqcup (-\ x \sqcup -\ (y \sqcup -\ (y \sqcup x))))) = y \sqcup -\ (y \sqcup x)$
using *3 4 966 1122* **by** *smt*
have *1234*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ (-\ x \sqcup (-\ x \sqcup (-\ y \sqcup -\ (y \sqcup -\ (x \sqcup y))))) = y \sqcup -\ (x \sqcup y)$
using *3 4 961 1122* **by** *metis*
have *1239*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ (-\ x \sqcup -\ y) = y \sqcup -\ (x \sqcup y)$
using *3 4 950 1122 1234* **by** *metis*
have *1240*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ (-\ y \sqcup -\ x) = y \sqcup -\ (y \sqcup x)$
using *3 4 948 1122 1230* **by** *metis*
have *1244*: $\bigwedge x\ y\ .\ x \sqcup -\ (x \sqcup (y \sqcup (y \sqcup -\ (x \sqcup y)))) = -\ (-\ x \sqcup -\ y) \sqcup -\ (y \sqcup (y \sqcup -\ (x \sqcup y)))$
using *3 4 944 1122 1167* **by** *metis*
have *1275*: $\bigwedge x\ y\ .\ x \sqcup (-\ y \sqcup (-\ (y \sqcup x) \sqcup -\ (x \sqcup (-\ x \sqcup (-\ y \sqcup -\ (y \sqcup x)))))) = x \sqcup -\ (y \sqcup x)$
using *10 800 1122* **by** *metis*
have *1346*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ (x \sqcup (y \sqcup (y \sqcup (x \sqcup -\ (x \sqcup (y \sqcup x)))))) = -\ x \sqcup -\ (x \sqcup y)$
using *3 4 10 266 1122 1167* **by** *smt*
have *1377*: $\bigwedge x\ y\ .\ -\ x \sqcup (y \sqcup -\ (-\ x \sqcup (y \sqcup (-\ x \sqcup -\ y)))) = y \sqcup -\ (x \sqcup y)$
using *78 1122 1239* **by** *metis*
have *1394*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcup -\ y) \sqcup -\ (y \sqcup (y \sqcup (-\ x \sqcup -\ (x \sqcup y)))) = x$
using *3 4 10 20 30 1122 1239* **by** *smt*
have *1427*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcup -\ y) \sqcup -\ (y \sqcup -\ (x \sqcup (x \sqcup -\ (x \sqcup y)))) = x \sqcup (x \sqcup -\ (x \sqcup y))$
using *3 4 30 40 1240* **by** *smt*
have *1436*: $\bigwedge x\ .\ -\ x \sqcup -\ (x \sqcup (x \sqcup (-\ x \sqcup -\ x))) = -\ x \sqcup (-\ x \sqcup -\ (x \sqcup -\ x))$
using *3 4 30 1140 1239* **by** *smt*

have *1437*: $\bigwedge x\ y\ .\ -\ (x \sqcup y) \sqcup -\ (x \sqcup -\ y) = -\ x$
using *6 1122* **by** *metis*
have *1438*: $\bigwedge x\ y\ .\ -\ (x \sqcup y) \sqcup -\ (y \sqcup -\ x) = -\ y$
using *12 1122* **by** *metis*
have *1439*: $\bigwedge x\ y\ .\ -\ (x \sqcup y) \sqcup -\ (-\ y \sqcup x) = -\ x$
using *13 1122* **by** *metis*
have *1440*: $\bigwedge x\ y\ .\ -\ (x \sqcup -\ y) \sqcup -\ (y \sqcup x) = -\ x$
using *20 1122* **by** *metis*
have *1441*: $\bigwedge x\ y\ .\ -\ (x \sqcup y) \sqcup -\ (-\ x \sqcup y) = -\ y$
using *21 1122* **by** *metis*
have *1568*: $\bigwedge x\ y\ .\ x \sqcup (-\ y \sqcup -\ x) = y \sqcup (-\ y \sqcup x)$
using *10 1122 1143* **by** *metis*
have *1598*: $\bigwedge x\ .\ -\ x \sqcup -\ (x \sqcup (x \sqcup (x \sqcup -\ x))) = -\ x \sqcup (-\ x \sqcup -\ (x \sqcup -\ x))$
using *4 1436 1568* **by** *metis*
have *1599*: $\bigwedge x\ y\ .\ -\ x \sqcup (y \sqcup -\ (x \sqcup (-\ x \sqcup (-\ x \sqcup y)))) = y \sqcup -\ (x \sqcup y)$
using *10 1377 1568* **by** *smt*
have *1617*: $\bigwedge x\ .\ x \sqcup (-\ x \sqcup (-\ x \sqcup -\ (x \sqcup -\ x))) = x \sqcup -\ x$
using *3 4 10 71 1122 1155 1568 1598* **by** *metis*
have *1632*: $\bigwedge x\ y\ z\ .\ -\ (x \sqcup -\ x) \sqcup -\ (-\ y \sqcup (-\ (z \sqcup -\ z) \sqcup -\ (y \sqcup -\ (x \sqcup -\ x)))) = y \sqcup -\ (x \sqcup -\ x)$
using *43 1157* **by** *metis*
have *1633*: $\bigwedge x\ y\ z\ .\ -\ (x \sqcup -\ x) \sqcup -\ (-\ y \sqcup (-\ (x \sqcup -\ x) \sqcup -\ (y \sqcup -\ (z \sqcup -\ z)))) = y \sqcup -\ (x \sqcup -\ x)$
using *43 1157* **by** *metis*
have *1636*: $\bigwedge x\ y\ .\ x \sqcup -\ (y \sqcup (-\ y \sqcup -\ (x \sqcup x))) = x \sqcup x$
using *43 1109 1122* **by** *metis*
have *1645*: $\bigwedge x\ y\ .\ x \sqcup -\ x = y \sqcup (y \sqcup -\ y)$
using *3 1110 1156* **by** *metis*
have *1648*: $\bigwedge x\ y\ z\ .\ -\ (x \sqcup (y \sqcup (-\ y \sqcup -\ x))) \sqcup -\ (z \sqcup -\ z) = -\ (y \sqcup -\ y)$
using *3 1115 1156* **by** *metis*
have *1657*: $\bigwedge x\ y\ z\ .\ x \sqcup -\ x = y \sqcup (z \sqcup -\ z)$
using *1105 1645* **by** *metis*
have *1664*: $\bigwedge x\ y\ z\ .\ x \sqcup -\ x = y \sqcup (z \sqcup -\ y)$
using *1115 1645* **by** *metis*
have *1672*: $\bigwedge x\ y\ z\ .\ x \sqcup -\ x = y \sqcup (-\ y \sqcup z)$
using *3 4 1657* **by** *metis*
have *1697*: $\bigwedge x\ y\ z\ .\ -\ x \sqcup (y \sqcup x) = z \sqcup -\ z$
using *1122 1664* **by** *metis*
have *1733*: $\bigwedge x\ y\ z\ .\ -\ (x \sqcup y) \sqcup -\ (-\ (z \sqcup -\ z) \sqcup -\ (-\ (-\ x \sqcup -\ x) \sqcup y)) = x \sqcup -\ x$
using *4 47 1105 1122* **by** *metis*
have *1791*: $\bigwedge x\ y\ z\ .\ x \sqcup -\ (y \sqcup (-\ y \sqcup z)) = x \sqcup -\ (x \sqcup -\ x)$
using *4 71 1122 1672* **by** *metis*
have *1818*: $\bigwedge x\ y\ z\ .\ x \sqcup -\ (-\ y \sqcup (z \sqcup y)) = x \sqcup -\ (x \sqcup -\ x)$
using *4 71 1122 1697* **by** *metis*
have *1861*: $\bigwedge x\ y\ z\ .\ -\ (x \sqcup -\ x) \sqcup -\ (y \sqcup -\ (z \sqcup -\ z)) = -\ y$
using *1437 1657* **by** *metis*
have *1867*: $\bigwedge x\ y\ z\ .\ -\ (x \sqcup -\ x) \sqcup -\ (-\ y \sqcup -\ (z \sqcup y)) = y$

using *1122 1437 1697* **by** *metis*
have *1868*: $\bigwedge x\ y\ .\ x \sqcup -\ (y \sqcup -\ y) = x$
using *1122 1155 1633 1861* **by** *metis*
have *1869*: $\bigwedge x\ y\ z\ .\ -\ (x \sqcup -\ x) \sqcup -\ (-\ y \sqcup (-\ (z \sqcup -\ z) \sqcup -\ y)) = y$
using *1632 1868* **by** *metis*
have *1870*: $\bigwedge x\ y\ .\ -\ (x \sqcup -\ x) \sqcup -\ y = -\ y$
using *1861 1868* **by** *metis*
have *1872*: $\bigwedge x\ y\ z\ .\ x \sqcup -\ (-\ y \sqcup (z \sqcup y)) = x$
using *1818 1868* **by** *metis*
have *1875*: $\bigwedge x\ y\ z\ .\ x \sqcup -\ (y \sqcup (-\ y \sqcup z)) = x$
using *1791 1868* **by** *metis*
have *1883*: $\bigwedge x\ y\ .\ -\ (x \sqcup (y \sqcup (-\ y \sqcup -\ x))) = -\ (y \sqcup -\ y)$
using *1648 1868* **by** *metis*
have *1885*: $\bigwedge x\ .\ x \sqcup (x \sqcup -\ x) = x \sqcup -\ x$
using *4 1568 1617 1868* **by** *metis*
have *1886*: $\bigwedge x\ .\ -\ x \sqcup -\ x = -\ x$
using *1598 1868 1885* **by** *metis*
have *1890*: $\bigwedge x\ .\ -\ (x \sqcup x) = -\ x$
using *1156 1868* **by** *metis*
have *1892*: $\bigwedge x\ y\ .\ -\ (x \sqcup -\ x) \sqcup y = y$
using *1122 1869 1870 1886* **by** *metis*
have *1893*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcup -\ (y \sqcup x)) = x$
using *1867 1892* **by** *metis*
have *1902*: $\bigwedge x\ y\ .\ x \sqcup (y \sqcup -\ (x \sqcup y)) = x \sqcup -\ x$
using *3 4 1122 1733 1886 1892* **by** *metis*
have *1908*: $\bigwedge x\ .\ x \sqcup x = x$
using *1636 1875 1890* **by** *metis*
have *1910*: $\bigwedge x\ y\ .\ x \sqcup -\ (y \sqcup x) = -\ y \sqcup x$
using *1599 1875* **by** *metis*
have *1921*: $\bigwedge x\ y\ .\ x \sqcup (-\ y \sqcup -\ (y \sqcup x)) = -\ y \sqcup x$
using *1275 1875 1910* **by** *metis*
have *1951*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ (y \sqcup x) = -\ x$
using *1227 1872 1893 1908* **by** *metis*
have *1954*: $\bigwedge x\ y\ z\ .\ x \sqcup (y \sqcup -\ (x \sqcup z)) = y \sqcup (-\ z \sqcup x)$
using *745 1122 1910 1951* **by** *metis*
have *1956*: $\bigwedge x\ y\ z\ .\ x \sqcup (-\ (x \sqcup y) \sqcup z) = -\ y \sqcup (x \sqcup z)$
using *678 1122 1910 1951* **by** *metis*
have *1959*: $\bigwedge x\ y\ .\ x \sqcup -\ (x \sqcup y) = -\ y \sqcup x$
using *86 1122 1910 1951* **by** *metis*
have *1972*: $\bigwedge x\ y\ .\ x \sqcup (-\ x \sqcup y) = x \sqcup -\ x$
using *1902 1910* **by** *metis*
have *2000*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcup -\ y) \sqcup -\ (y \sqcup (-\ x \sqcup y)) = x \sqcup -\ (y \sqcup (-\ x \sqcup y))$
using *4 1244 1910 1959* **by** *metis*
have *2054*: $\bigwedge x\ y\ .\ x \sqcup -\ (y \sqcup (-\ x \sqcup y)) = x$
using *1394 1921 2000* **by** *metis*
have *2057*: $\bigwedge x\ y\ .\ -\ (x \sqcup (y \sqcup -\ y)) = -\ (y \sqcup -\ y)$
using *1883 1972* **by** *metis*
have *2061*: $\bigwedge x\ y\ .\ x \sqcup (-\ y \sqcup x) = x \sqcup -\ y$

using *4 1122 1427 1910 1959 2054* **by** *metis*
have *2090*: $\bigwedge x\ y\ z\ .\ x \sqcup (-\ (y \sqcup x) \sqcup z) = x \sqcup (-\ y \sqcup z)$
using *1122 1169 1956* **by** *metis*
have *2100*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ (x \sqcup y) = -\ x$
using *4 1346 1868 1885 1910 1959 1972 2057* **by** *metis*
have *2144*: $\bigwedge x\ y\ .\ x \sqcup -\ (y \sqcup -\ x) = x$
using *1122 1440 2000 2061* **by** *metis*
have *2199*: $\bigwedge x\ y\ .\ x \sqcup (x \sqcup y) = x \sqcup y$
using *3 1908* **by** *metis*
have *2208*: $\bigwedge x\ y\ z\ .\ x \sqcup (-\ (y \sqcup -\ x) \sqcup z) = x \sqcup z$
using *3 2144* **by** *metis*
have *2349*: $\bigwedge x\ y\ z\ .\ -\ (x \sqcup y) \sqcup -\ (x \sqcup (y \sqcup z)) = -\ (x \sqcup y)$
using *3 2100* **by** *metis*
have *2432*: $\bigwedge x\ y\ z\ .\ -\ (x \sqcup (y \sqcup z)) \sqcup -\ (y \sqcup (z \sqcup -\ x)) = -\ (y \sqcup z)$
using *3 1438* **by** *metis*
have *2530*: $\bigwedge x\ y\ z\ .\ -\ (-\ (x \sqcup y) \sqcup z) = -\ (y \sqcup (-\ x \sqcup z)) \sqcup -\ (-\ y \sqcup z)$
using *4 1122 1439 2090 2208* **by** *smt*
have *3364*: $\bigwedge x\ y\ z\ .\ -\ (-\ x \sqcup y) \sqcup (z \sqcup -\ (x \sqcup y)) = z \sqcup -\ y$
using *3 4 1122 1441 1910 1954 2199* **by** *metis*
have *5763*: $\bigwedge x\ y\ z\ .\ -\ (x \sqcup y) \sqcup -\ (-\ x \sqcup (y \sqcup z)) = -\ (x \sqcup y) \sqcup -\ (y \sqcup z)$
using *4 2349 3364* **by** *metis*
have *6113*: $\bigwedge x\ y\ z\ .\ -\ (x \sqcup (y \sqcup z)) \sqcup -\ (z \sqcup -\ x) = -\ (y \sqcup z) \sqcup -\ (z \sqcup -$
$x)$
using *4 2432 3364 5763* **by** *metis*
show $\bigwedge x\ y\ z.\ x \sqcup y \sqcap z = (x \sqcup y) \sqcap (x \sqcup z)$
proof $-$
fix $x\ y\ z$
have $-\ (y \sqcap z \sqcup x) = -\ (-\ (-\ y \sqcup z) \sqcup -\ (-\ y \sqcup -\ z) \sqcup x) \sqcup -\ (x \sqcup -$
$-\ z)$
using *1437 2530 6113* **by** (*smt commutative inf-def*)
thus $x \sqcup y \sqcap z = (x \sqcup y) \sqcap (x \sqcup z)$
using *12 1122* **by** (*metis commutative inf-def*)
qed
qed
show *14*: $\bigwedge x.\ x \sqcap -\ x = bot$
proof $-$
fix x
have $(bot \sqcup x) \sqcap (bot \sqcup -x) = bot$
using *huntington bot-def inf-def* **by** *auto*
thus $x \sqcap -x = bot$
using *11 less-eq-def* **by** *force*
qed
show *15*: $\bigwedge x.\ x \sqcup -\ x = top$
using *5 14* **by** (*metis* (*no-types*, *lifting*) *huntington bot-def less-eq-def top-def*)
show *16*: $\bigwedge x\ y.\ x - y = x \sqcap -\ y$
using *15* **by** (*metis commutative huntington inf-def minus-def*)
show *7*: $\bigwedge x\ y\ z.\ x \leq y \Longrightarrow x \leq z \Longrightarrow x \leq y \sqcap z$
by (*simp add*: *13 less-eq-def*)
qed

end

context *boolean-algebra*
begin

sublocale *ba-he*: *huntington-extended*
proof
  show $\bigwedge x\ y\ z.\ x \sqcup (y \sqcup z) = x \sqcup y \sqcup z$
    by (*simp add*: *sup-assoc*)
  show $\bigwedge x\ y.\ x \sqcup y = y \sqcup x$
    by (*simp add*: *sup-commute*)
  show $\bigwedge x\ y.\ x = -\ (-\ x \sqcup y) \sqcup -\ (-\ x \sqcup -\ y)$
    by *simp*
  show $top = (THE\ x.\ \forall\ y.\ x = y \sqcup -\ y)$
    by *auto*
  show $bot = -\ (THE\ x.\ \forall\ y.\ x = y \sqcup -\ y)$
    by *auto*
  show $\bigwedge x\ y.\ x \sqcap y = -\ (-\ x \sqcup -\ y)$
    by *simp*
  show $\bigwedge x\ y.\ x - y = -\ (-\ x \sqcup y)$
    by (*simp add*: *diff-eq*)
  show $\bigwedge x\ y.\ (x \leq y) = (x \sqcup y = y)$
    by (*simp add*: *le-iff-sup*)
  show $\bigwedge x\ y.\ (x < y) = (x \sqcup y = y \wedge y \sqcup x \neq x)$
    using *sup.strict-order-iff sup-commute* **by** *auto*
qed

end

2.3 Stone Algebras

We relate Stone algebras to Boolean algebras.

class *stone-algebra-extended* = *stone-algebra* + *minus* +
  assumes *stone-minus-def*[*simp*]: $x - y = x \sqcap -y$

class *regular-stone-algebra* = *stone-algebra-extended* +
  assumes *double-complement*[*simp*]: $--x = x$
begin

subclass *boolean-algebra*
proof
  show $\bigwedge x.\ x \sqcap -\ x = bot$
    by *simp*
  show $\bigwedge x.\ x \sqcup -\ x = top$
    using *regular-dense-top* **by** *fastforce*
  show $\bigwedge x\ y.\ x - y = x \sqcap -\ y$
    by *simp*
qed

end

context *boolean-algebra*
begin

sublocale *ba-rsa*: *regular-stone-algebra*
proof
  show $\bigwedge x\ y.\ x - y = x \sqcap - y$
    by (*simp add*: *diff-eq*)
  show $\bigwedge x.\ - - x = x$
    by *simp*
qed

end

3 Alternative Axiomatisations of Boolean Algebras

We consider four axiomatisations of Boolean algebras based only on join and complement. The first three are from the literature and the fourth, a version using equational axioms, is new. The motivation for Byrne's and the new axiomatisation is that the axioms are easier to understand than Huntington's third axiom. We also include Meredith's axiomatisation.

3.1 Lee Byrne's Formulation A

The following axiomatisation is from [2, Formulation A]; see also [10].

Theorem 3

class *boolean-algebra-1* = *sup* + *uminus* +
  assumes *ba1-associative*: $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$
  assumes *ba1-commutative*: $x \sqcup y = y \sqcup x$
  assumes *ba1-complement*: $x \sqcup -y = z \sqcup -z \longleftrightarrow x \sqcup y = x$
begin

subclass *huntington*
proof
  show *1*: $\bigwedge x\ y\ z.\ x \sqcup (y \sqcup z) = x \sqcup y \sqcup z$
    by (*simp add*: *ba1-associative*)
  show $\bigwedge x\ y.\ x \sqcup y = y \sqcup x$
    by (*simp add*: *ba1-commutative*)
  show $\bigwedge x\ y.\ x = - (- x \sqcup y) \sqcup - (- x \sqcup - y)$
  proof −
    have *2*: $\forall x\ y.\ y \sqcup (y \sqcup x) = y \sqcup x$
      using *1* **by** (*metis ba1-complement*)
    hence $\forall x.\ --x = x$

```
      by (smt ba1-associative ba1-commutative ba1-complement)
    hence ∀ x y. y ⊔ −(y ⊔ −x) = y ⊔ x
      by (smt ba1-associative ba1-commutative ba1-complement)
    thus ⋀x y. x = −(−x ⊔ y) ⊔ −(−x ⊔ − y)
      using 2 by (smt ba1-commutative ba1-complement)
  qed
qed

end

context huntington
begin

sublocale h-ba1: boolean-algebra-1
proof
  show ⋀x y z. x ⊔ (y ⊔ z) = x ⊔ y ⊔ z
    by (simp add: associative)
  show ⋀x y. x ⊔ y = y ⊔ x
    by (simp add: commutative)
  show ⋀x y z. (x ⊔ − y = z ⊔ − z) = (x ⊔ y = x)
  proof
    fix x y z
    have 1: ⋀x y z. −(−x ⊔ y) ⊔ (−(−x ⊔ −y) ⊔ z) = x ⊔ z
      using associative huntington by force
    have 2: ⋀x y. −(x ⊔ −y) ⊔ −(−y ⊔ −x) = y
      by (metis commutative huntington)
    show x ⊔ − y = z ⊔ − z ⟹ x ⊔ y = x
      by (metis 1 2 associative commutative top-unique)
    show x ⊔ y = x ⟹ x ⊔ − y = z ⊔ − z
      by (metis associative huntington commutative top-unique)
  qed
qed

end
```

3.2 Lee Byrne's Formulation B

The following axiomatisation is from [2, Formulation B].

Theorem 4

```
class boolean-algebra-2 = sup + uminus +
  assumes ba2-associative-commutative: (x ⊔ y) ⊔ z = (y ⊔ z) ⊔ x
  assumes ba2-complement: x ⊔ −y = z ⊔ −z ⟷ x ⊔ y = x
begin

subclass boolean-algebra-1
proof
  show ⋀x y z. x ⊔ (y ⊔ z) = x ⊔ y ⊔ z
    by (smt ba2-associative-commutative ba2-complement)
```

show $\bigwedge x\ y.\ x \sqcup y = y \sqcup x$
  by (*metis ba2-associative-commutative ba2-complement*)
show $\bigwedge x\ y\ z.\ (x \sqcup -\ y = z \sqcup -\ z) = (x \sqcup y = x)$
  by (*simp add*: *ba2-complement*)
qed

end

context *boolean-algebra-1*
begin

sublocale *ba1-ba2*: *boolean-algebra-2*
proof
show $\bigwedge x\ y\ z.\ x \sqcup y \sqcup z = y \sqcup z \sqcup x$
  using *ba1-associative commutative* **by** *force*
show $\bigwedge x\ y\ z.\ (x \sqcup -\ y = z \sqcup -\ z) = (x \sqcup y = x)$
  by (*simp add*: *ba1-complement*)
qed

end

3.3 Meredith's Equational Axioms

The following axiomatisation is from [22, page 221 (1) {A,N}].

class *boolean-algebra-mp* = *sup* + *uminus* +
assumes *ba-mp-1*: $-(-x \sqcup y) \sqcup x = x$
assumes *ba-mp-2*: $-(-x \sqcup y) \sqcup (z \sqcup y) = y \sqcup (z \sqcup x)$
begin

subclass *huntington*
proof
show $\bigwedge x\ y\ z.\ x \sqcup (y \sqcup z) = x \sqcup y \sqcup z$
  by (*metis ba-mp-1 ba-mp-2*)
show $\bigwedge x\ y.\ x \sqcup y = y \sqcup x$
  by (*metis ba-mp-1 ba-mp-2*)
show $\bigwedge x\ y.\ x = -\ (-\ x \sqcup y) \sqcup -\ (-\ x \sqcup -\ y)$
  by (*metis ba-mp-1 ba-mp-2*)
qed

end

context *huntington*
begin

sublocale *mp-h*: *boolean-algebra-mp*
proof
show *1*: $\bigwedge x\ y.\ -\ (-\ x \sqcup y) \sqcup x = x$
  by (*metis h-ba1.ba1-associative h-ba1.ba1-complement huntington*)
show $\bigwedge x\ y\ z.\ -\ (-\ x \sqcup y) \sqcup (z \sqcup y) = y \sqcup (z \sqcup x)$

proof –
fix $x\ y\ z$
have $y = -(-x \sqcup -y) \sqcup y$
using *1 h-ba1.ba1-commutative* **by** *auto*
thus $-(-x \sqcup y) \sqcup (z \sqcup y) = y \sqcup (z \sqcup x)$
by (*metis h-ba1.ba1-associative h-ba1.ba1-commutative huntington*)
qed
qed

end

3.4 An Equational Axiomatisation based on Semilattices

The following version is an equational axiomatisation based on semilattices. We add the double complement rule and that *top* is unique. The final axiom *ba3-export* encodes the logical statement $P \vee Q = P \vee (\neg P \wedge Q)$. Its dual appears in [1].

Theorem 5

class *boolean-algebra-3* = *sup* + *uminus* +
assumes *ba3-associative*: $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$
assumes *ba3-commutative*: $x \sqcup y = y \sqcup x$
assumes *ba3-idempotent*[*simp*]: $x \sqcup x = x$
assumes *ba3-double-complement*[*simp*]: $--x = x$
assumes *ba3-top-unique*: $x \sqcup -x = y \sqcup -y$
assumes *ba3-export*: $x \sqcup -(x \sqcup y) = x \sqcup -y$
begin

subclass *huntington*
proof
show $\bigwedge x\ y\ z.\ x \sqcup (y \sqcup z) = x \sqcup y \sqcup z$
by (*simp add*: *ba3-associative*)
show $\bigwedge x\ y.\ x \sqcup y = y \sqcup x$
by (*simp add*: *ba3-commutative*)
show $\bigwedge x\ y.\ x = -(-x \sqcup y) \sqcup -(-x \sqcup -y)$
by (*metis ba3-commutative ba3-double-complement ba3-export ba3-idempotent ba3-top-unique*)
qed

end

context *huntington*
begin

sublocale *h-ba3*: *boolean-algebra-3*
proof
show $\bigwedge x\ y\ z.\ x \sqcup (y \sqcup z) = x \sqcup y \sqcup z$
by (*simp add*: *h-ba1.ba1-associative*)
show $\bigwedge x\ y.\ x \sqcup y = y \sqcup x$

by (*simp add*: *h-ba1.ba1-commutative*)
show *3*: $\bigwedge x.\ x \sqcup x = x$
using *h-ba1.ba1-complement* **by** *blast*
show *4*: $\bigwedge x.\ - - x = x$
by (*metis h-ba1.ba1-commutative huntington top-unique*)
show $\bigwedge x\ y.\ x \sqcup - x = y \sqcup - y$
by (*simp add*: *top-unique*)
show $\bigwedge x\ y.\ x \sqcup - (x \sqcup y) = x \sqcup - y$
using *3 4* **by** (*smt h-ba1.ba1-ba2.ba2-associative-commutative h-ba1.ba1-complement*)
qed

end

4 Subset Boolean Algebras

We apply Huntington's axioms to the range of a unary operation, which serves as complement on the range. This gives a Boolean algebra structure on the range without imposing any further constraints on the set. The obtained structure is used as a reference in the subsequent development and to inherit the results proved here. This is taken from [13, 16] and follows the development of Boolean algebras in [20].

Definition 6

class *subset-boolean-algebra* = *sup* + *uminus* +
assumes *sub-associative*: $-x \sqcup (-y \sqcup -z) = (-x \sqcup -y) \sqcup -z$
assumes *sub-commutative*: $-x \sqcup -y = -y \sqcup -x$
assumes *sub-complement*: $-x = -(--x \sqcup -y) \sqcup -(--x \sqcup --y)$
assumes *sub-sup-closed*: $-x \sqcup -y = --(-x \sqcup -y)$
begin

uniqueness of *top*, resulting in the lemma *top-def* to replace the assumption *sub-top-def*

lemma *top-unique*:
$-x \sqcup --x = -y \sqcup --y$
by (*metis sub-associative sub-commutative sub-complement*)

consequences for join and complement

lemma *double-negation*[*simp*]:
$---x = -x$
by (*metis sub-complement sub-sup-closed*)

lemma *complement-1*:
$--x = -(-x \sqcup -y) \sqcup -(-x \sqcup --y)$
by (*metis double-negation sub-complement*)

lemma *sup-right-zero-var*:
$-x \sqcup (-y \sqcup --y) = -z \sqcup --z$

by (*smt complement-1 sub-associative sub-sup-closed top-unique*)

lemma *sup-right-unit-idempotent*:
$-x \sqcup -x = -x \sqcup -(-y \sqcup --y)$
by (*metis complement-1 double-negation sub-sup-closed sup-right-zero-var*)

lemma *sup-idempotent*[*simp*]:
$-x \sqcup -x = -x$
by (*smt complement-1 double-negation sub-associative sup-right-unit-idempotent*)

lemma *complement-2*:
$-x = -(-(-x \sqcup -y) \sqcup -(-x \sqcup --y))$
using *complement-1* **by** *auto*

lemma *sup-eq-cases*:
$-x \sqcup -y = -x \sqcup -z \Longrightarrow --x \sqcup -y = --x \sqcup -z \Longrightarrow -y = -z$
by (*metis complement-2 sub-commutative*)

lemma *sup-eq-cases-2*:
$-y \sqcup -x = -z \sqcup -x \Longrightarrow -y \sqcup --x = -z \sqcup --x \Longrightarrow -y = -z$
using *sub-commutative sup-eq-cases* **by** *auto*

end

Definition 7

class *subset-extended* = *sup* + *inf* + *minus* + *uminus* + *bot* + *top* + *ord* +
assumes *sub-top-def*: $top = (THE\ x\ .\ \forall y\ .\ x = -y \sqcup --y)$
assumes *sub-bot-def*: $bot = -(THE\ x\ .\ \forall y\ .\ x = -y \sqcup --y)$
assumes *sub-inf-def*: $-x \sqcap -y = -(--x \sqcup --y)$
assumes *sub-minus-def*: $-x - -y = -(--x \sqcup -y)$
assumes *sub-less-eq-def*: $-x \le -y \longleftrightarrow -x \sqcup -y = -y$
assumes *sub-less-def*: $-x < -y \longleftrightarrow -x \sqcup -y = -y \wedge \neg (-y \sqcup -x = -x)$

class *subset-boolean-algebra-extended* = *subset-boolean-algebra* + *subset-extended*
begin

lemma *top-def*:
$top = -x \sqcup --x$
using *sub-top-def top-unique* **by** *blast*

consequences for meet

lemma *inf-closed*:
$-x \sqcap -y = --(-x \sqcap -y)$
by (*simp add*: *sub-inf-def*)

lemma *inf-associative*:
$-x \sqcap (-y \sqcap -z) = (-x \sqcap -y) \sqcap -z$
using *sub-associative sub-inf-def sub-sup-closed* **by** *auto*

lemma *inf-commutative*:
$-x \sqcap -y = -y \sqcap -x$
by (*simp add*: *sub-commutative sub-inf-def*)

lemma *inf-idempotent*[*simp*]:
$-x \sqcap -x = -x$
by (*simp add*: *sub-inf-def*)

lemma *inf-absorb*[*simp*]:
$(-x \sqcup -y) \sqcap -x = -x$
by (*metis complement-1 sup-idempotent sub-inf-def sub-associative sub-sup-closed*)

lemma *sup-absorb*[*simp*]:
$-x \sqcup (-x \sqcap -y) = -x$
by (*metis sub-associative sub-complement sub-inf-def sup-idempotent*)

lemma *inf-demorgan*:
$-(-x \sqcap -y) = --x \sqcup --y$
using *sub-inf-def sub-sup-closed* **by** *auto*

lemma *sub-sup-demorgan*:
$-(-x \sqcup -y) = --x \sqcap --y$
by (*simp add*: *sub-inf-def*)

lemma *sup-cases*:
$-x = (-x \sqcap -y) \sqcup (-x \sqcap --y)$
by (*metis inf-closed inf-demorgan sub-complement*)

lemma *inf-cases*:
$-x = (-x \sqcup -y) \sqcap (-x \sqcup --y)$
by (*metis complement-2 sub-sup-closed sub-sup-demorgan*)

lemma *inf-complement-intro*:
$(-x \sqcup -y) \sqcap --x = -y \sqcap --x$
proof −
have $(-x \sqcup -y) \sqcap --x = (-x \sqcup -y) \sqcap (--x \sqcup -y) \sqcap --x$
by (*metis inf-absorb inf-associative sub-sup-closed*)
also have ... $= -y \sqcap --x$
by (*metis inf-cases sub-commutative*)
finally show *?thesis*
.
qed

lemma *sup-complement-intro*:
$-x \sqcup -y = -x \sqcup (--x \sqcap -y)$
by (*metis inf-absorb inf-commutative inf-complement-intro sub-sup-closed sup-cases*)

lemma *inf-left-dist-sup*:
  $-x \sqcap (-y \sqcup -z) = (-x \sqcap -y) \sqcup (-x \sqcap -z)$
proof −
  have $-x \sqcap (-y \sqcup -z) = (-x \sqcap (-y \sqcup -z) \sqcap -y) \sqcup (-x \sqcap (-y \sqcup -z) \sqcap --y)$
    by (*metis sub-inf-def sub-sup-closed sup-cases*)
  also have ... $= (-x \sqcap -y) \sqcup (-x \sqcap -z \sqcap --y)$
    by (*metis inf-absorb inf-associative inf-complement-intro sub-sup-closed*)
  also have ... $= (-x \sqcap -y) \sqcup ((-x \sqcap -y \sqcap -z) \sqcup (-x \sqcap -z \sqcap --y))$
    using *sub-associative sub-inf-def sup-absorb* **by** *auto*
  also have ... $= (-x \sqcap -y) \sqcup ((-x \sqcap -z \sqcap -y) \sqcup (-x \sqcap -z \sqcap --y))$
    by (*metis inf-associative inf-commutative*)
  also have ... $= (-x \sqcap -y) \sqcup (-x \sqcap -z)$
    by (*metis sub-inf-def sup-cases*)
  finally show *?thesis*
    .
qed

lemma *sup-left-dist-inf*:
  $-x \sqcup (-y \sqcap -z) = (-x \sqcup -y) \sqcap (-x \sqcup -z)$
proof −
  have $-x \sqcup (-y \sqcap -z) = -(--x \sqcap (--y \sqcup --z))$
    by (*metis sub-inf-def sub-sup-closed sub-sup-demorgan*)
  also have ... $= (-x \sqcup -y) \sqcap (-x \sqcup -z)$
    by (*metis inf-left-dist-sup sub-sup-closed sub-sup-demorgan*)
  finally show *?thesis*
    .
qed

lemma *sup-right-dist-inf*:
  $(-y \sqcap -z) \sqcup -x = (-y \sqcup -x) \sqcap (-z \sqcup -x)$
  using *sub-commutative sub-inf-def sup-left-dist-inf* **by** *auto*

lemma *inf-right-dist-sup*:
  $(-y \sqcup -z) \sqcap -x = (-y \sqcap -x) \sqcup (-z \sqcap -x)$
  by (*metis inf-commutative inf-left-dist-sup sub-sup-closed*)

lemma *case-duality*:
  $(--x \sqcap -y) \sqcup (-x \sqcap -z) = (-x \sqcup -y) \sqcap (--x \sqcup -z)$
proof −
  have *1*: $-(--x \sqcap --y) \sqcap ----x = --x \sqcap -y$
    using *inf-commutative inf-complement-intro sub-sup-closed sub-sup-demorgan*
by *auto*
  have *2*: $-(----x \sqcup -(--x \sqcup -z)) = -----x \sqcap ---z$
    by (*metis* (*no-types*) *double-negation sup-complement-intro sub-sup-demorgan*)
  have *3*: $-(--x \sqcap --y) \sqcap -x = -x$
    using *inf-commutative inf-left-dist-sup sub-sup-closed sub-sup-demorgan* **by**
auto
  hence $-(--x \sqcap --y) = -x \sqcup -y$

using *sub-sup-closed sub-sup-demorgan* **by** *auto*
thus *?thesis*
by (*metis double-negation 1 2 3 inf-associative inf-left-dist-sup sup-complement-intro*)
qed

lemma *case-duality-2*:
$(-x \sqcap -y) \sqcup (--x \sqcap -z) = (-x \sqcup -z) \sqcap (--x \sqcup -y)$
using *case-duality sub-commutative sub-inf-def* **by** *auto*

lemma *complement-cases*:
$((-v \sqcap -w) \sqcup (--v \sqcap -x)) \sqcap -((-v \sqcap -y) \sqcup (--v \sqcap -z)) = (-v \sqcap -w \sqcap --y) \sqcup (--v \sqcap -x \sqcap --z)$
proof −
have *1*: $(--v \sqcup -w) = --(--v \sqcup -w) \wedge (-v \sqcup -x) = --(-v \sqcup -x) \wedge (--v \sqcup --y) = --(--v \sqcup --y) \wedge (-v \sqcup --z) = --(-v \sqcup --z)$
using *sub-inf-def sub-sup-closed* **by** *auto*
have *2*: $(-v \sqcup (-x \sqcap --z)) = --(-v \sqcup (-x \sqcap --z))$
using *sub-inf-def sub-sup-closed* **by** *auto*
have $((-v \sqcap -w) \sqcup (--v \sqcap -x)) \sqcap -((-v \sqcap -y) \sqcup (--v \sqcap -z)) = ((-v \sqcap -w) \sqcup (--v \sqcap -x)) \sqcap (-(-v \sqcap -y) \sqcap -(--v \sqcap -z))$
using *sub-inf-def* **by** *auto*
also have ... $= ((-v \sqcap -w) \sqcup (--v \sqcap -x)) \sqcap ((--v \sqcup --y) \sqcap (-v \sqcup --z))$
using *inf-demorgan* **by** *auto*
also have ... $= (--v \sqcup -w) \sqcap (-v \sqcup -x) \sqcap ((--v \sqcup --y) \sqcap (-v \sqcup --z))$
by (*metis case-duality double-negation*)
also have ... $= (--v \sqcup -w) \sqcap ((-v \sqcup -x) \sqcap ((--v \sqcup --y) \sqcap (-v \sqcup --z)))$
by (*metis 1 inf-associative sub-inf-def*)
also have ... $= (--v \sqcup -w) \sqcap ((-v \sqcup -x) \sqcap (--v \sqcup --y) \sqcap (-v \sqcup --z))$
by (*metis 1 inf-associative*)
also have ... $= (--v \sqcup -w) \sqcap ((--v \sqcup --y) \sqcap (-v \sqcup -x) \sqcap (-v \sqcup --z))$
by (*metis 1 inf-commutative*)
also have ... $= (--v \sqcup -w) \sqcap ((--v \sqcup --y) \sqcap ((-v \sqcup -x) \sqcap (-v \sqcup --z)))$
by (*metis 1 inf-associative*)
also have ... $= (--v \sqcup -w) \sqcap ((--v \sqcup --y) \sqcap (-v \sqcup (-x \sqcap --z)))$
by (*simp add*: *sup-left-dist-inf*)
also have ... $= (--v \sqcup -w) \sqcap (--v \sqcup --y) \sqcap (-v \sqcup (-x \sqcap --z))$
using *1 2* **by** (*metis inf-associative*)
also have ... $= (--v \sqcup (-w \sqcap --y)) \sqcap (-v \sqcup (-x \sqcap --z))$
by (*simp add*: *sup-left-dist-inf*)
also have ... $= (-v \sqcap (-w \sqcap --y)) \sqcup (--v \sqcap (-x \sqcap --z))$
by (*metis case-duality complement-1 complement-2 sub-inf-def*)
also have ... $= (-v \sqcap -w \sqcap --y) \sqcup (--v \sqcap -x \sqcap --z)$
by (*simp add*: *inf-associative*)
finally show *?thesis*
.
qed

lemma *inf-cases-2*: $--x = -(-x \sqcap -y) \sqcap -(-x \sqcap --y)$

using *sub-inf-def sup-cases* **by** *auto*

consequences for *top* and *bot*

lemma *sup-complement*[*simp*]:
$-x \sqcup --x = top$
using *top-def* **by** *auto*

lemma *inf-complement*[*simp*]:
$-x \sqcap --x = bot$
by (*metis sub-bot-def sub-inf-def sub-top-def top-def*)

lemma *complement-bot*[*simp*]:
$-bot = top$
using *inf-complement inf-demorgan sup-complement* **by** *fastforce*

lemma *complement-top*[*simp*]:
$-top = bot$
using *sub-bot-def sub-top-def* **by** *blast*

lemma *sup-right-zero*[*simp*]:
$-x \sqcup top = top$
using *sup-right-zero-var* **by** *auto*

lemma *sup-left-zero*[*simp*]:
$top \sqcup -x = top$
by (*metis complement-bot sub-commutative sup-right-zero*)

lemma *inf-right-unit*[*simp*]:
$-x \sqcap bot = bot$
by (*metis complement-bot complement-top double-negation sub-sup-demorgan sup-right-zero*)

lemma *inf-left-unit*[*simp*]:
$bot \sqcap -x = bot$
by (*metis complement-top inf-commutative inf-right-unit*)

lemma *sup-right-unit*[*simp*]:
$-x \sqcup bot = -x$
using *sup-right-unit-idempotent* **by** *auto*

lemma *sup-left-unit*[*simp*]:
$bot \sqcup -x = -x$
by (*metis complement-top sub-commutative sup-right-unit*)

lemma *inf-right-zero*[*simp*]:
$-x \sqcap top = -x$
by (*metis inf-left-dist-sup sup-cases top-def*)

lemma *sub-inf-left-zero*[*simp*]:

top $\sqcap -x = -x$
using *inf-absorb top-def* **by** *fastforce*

lemma *bot-double-complement*[*simp*]:
$--bot = bot$
by *simp*

lemma *top-double-complement*[*simp*]:
$--top = top$
by *simp*

consequences for the order

lemma *reflexive*:
$-x \leq -x$
by (*simp add*: *sub-less-eq-def*)

lemma *transitive*:
$-x \leq -y \Longrightarrow -y \leq -z \Longrightarrow -x \leq -z$
by (*metis sub-associative sub-less-eq-def*)

lemma *antisymmetric*:
$-x \leq -y \Longrightarrow -y \leq -x \Longrightarrow -x = -y$
by (*simp add*: *sub-commutative sub-less-eq-def*)

lemma *sub-bot-least*:
$bot \leq -x$
using *sup-left-unit complement-top sub-less-eq-def* **by** *blast*

lemma *top-greatest*:
$-x \leq top$
using *complement-bot sub-less-eq-def sup-right-zero* **by** *blast*

lemma *upper-bound-left*:
$-x \leq -x \sqcup -y$
by (*metis sub-associative sub-less-eq-def sub-sup-closed sup-idempotent*)

lemma *upper-bound-right*:
$-y \leq -x \sqcup -y$
using *sub-commutative upper-bound-left* **by** *fastforce*

lemma *sub-sup-left-isotone*:
assumes $-x \leq -y$
shows $-x \sqcup -z \leq -y \sqcup -z$
proof –
have $-x \sqcup -y = -y$
by (*meson assms sub-less-eq-def*)
thus *?thesis*
by (*metis* (*full-types*) *sub-associative sub-commutative sub-sup-closed upper-bound-left*)

qed

lemma *sub-sup-right-isotone*:
$-x \leq -y \Longrightarrow -z \sqcup -x \leq -z \sqcup -y$
by (*simp add*: *sub-commutative sub-sup-left-isotone*)

lemma *sup-isotone*:
assumes $-p \leq -q$
and $-r \leq -s$
shows $-p \sqcup -r \leq -q \sqcup -s$
proof –
have $\bigwedge x\ y.\ \neg -x \leq -y \sqcup -r \vee -x \leq -y \sqcup -s$
by (*metis* (*full-types*) *assms*(*2*) *sub-sup-closed sub-sup-right-isotone transitive*)
thus *?thesis*
by (*metis* (*no-types*) *assms*(*1*) *sub-sup-closed sub-sup-left-isotone*)
qed

lemma *sub-complement-antitone*:
$-x \leq -y \Longrightarrow --y \leq --x$
by (*metis inf-absorb inf-demorgan sub-less-eq-def*)

lemma *less-eq-inf*:
$-x \leq -y \longleftrightarrow -x \sqcap -y = -x$
by (*metis inf-absorb inf-commutative sub-less-eq-def upper-bound-right sup-absorb*)

lemma *inf-complement-left-antitone*:
$-x \leq -y \Longrightarrow -(-y \sqcap -z) \leq -(-x \sqcap -z)$
by (*simp add*: *sub-complement-antitone inf-demorgan sub-sup-left-isotone*)

lemma *sub-inf-left-isotone*:
$-x \leq -y \Longrightarrow -x \sqcap -z \leq -y \sqcap -z$
using *sub-complement-antitone inf-closed inf-complement-left-antitone* **by** *fastforce*

lemma *sub-inf-right-isotone*:
$-x \leq -y \Longrightarrow -z \sqcap -x \leq -z \sqcap -y$
by (*simp add*: *inf-commutative sub-inf-left-isotone*)

lemma *inf-isotone*:
assumes $-p \leq -q$
and $-r \leq -s$
shows $-p \sqcap -r \leq -q \sqcap -s$
proof –
have $\forall w\ x\ y\ z.\ (-w \leq -x \sqcap -y \vee \neg -w \leq -x \sqcap -z) \vee \neg -z \leq -y$
by (*metis* (*no-types*) *inf-closed sub-inf-right-isotone transitive*)
thus *?thesis*
by (*metis* (*no-types*) *assms inf-closed sub-inf-left-isotone*)
qed

lemma *least-upper-bound*:
$-x \le -z \wedge -y \le -z \longleftrightarrow -x \sqcup -y \le -z$
by (*metis sub-sup-closed transitive upper-bound-right sup-idempotent sup-isotone upper-bound-left*)

lemma *lower-bound-left*:
$-x \sqcap -y \le -x$
by (*metis sub-inf-def upper-bound-right sup-absorb*)

lemma *lower-bound-right*:
$-x \sqcap -y \le -y$
using *inf-commutative lower-bound-left* **by** *fastforce*

lemma *greatest-lower-bound*:
$-x \le -y \wedge -x \le -z \longleftrightarrow -x \le -y \sqcap -z$
by (*metis inf-closed sub-inf-left-isotone less-eq-inf transitive lower-bound-left lower-bound-right*)

lemma *less-eq-sup-top*:
$-x \le -y \longleftrightarrow --x \sqcup -y = top$
by (*metis complement-1 inf-commutative inf-complement-intro sub-inf-left-zero less-eq-inf sub-complement sup-complement-intro top-def*)

lemma *less-eq-inf-bot*:
$-x \le -y \longleftrightarrow -x \sqcap --y = bot$
by (*metis complement-bot complement-top double-negation inf-demorgan less-eq-sup-top sub-inf-def*)

lemma *shunting*:
$-x \sqcap -y \le -z \longleftrightarrow -y \le --x \sqcup -z$
proof (*cases* $--x \sqcup (-z \sqcup --y) = top$)
case *True*
have $\forall v\ w.\ -v \le -w \vee -w \sqcup --v \ne top$
using *less-eq-sup-top sub-commutative* **by** *blast*
thus *?thesis*
by (*metis True sub-associative sub-commutative sub-inf-def sub-sup-closed*)
next
case *False*
hence $--x \sqcup (-z \sqcup --y) \ne top \wedge \neg -y \le -z \sqcup --x$
by (*metis* (*no-types*) *less-eq-sup-top sub-associative sub-commutative sub-sup-closed*)
thus *?thesis*
using *less-eq-sup-top sub-associative sub-commutative sub-inf-def sub-sup-closed* **by** *auto*
qed

lemma *shunting-right*:
$-x \sqcap -y \le -z \longleftrightarrow -x \le -z \sqcup --y$

by (*metis inf-commutative sub-commutative shunting*)

lemma *sup-less-eq-cases*:
assumes $-z \leq -x \sqcup -y$
and $-z \leq --x \sqcup -y$
shows $-z \leq -y$
proof −
have $-z \leq (-x \sqcup -y) \sqcap (--x \sqcup -y)$
by (*metis assms greatest-lower-bound sub-sup-closed*)
also have ... $= -y$
by (*metis inf-cases sub-commutative*)
finally show *?thesis*
.
qed

lemma *sup-less-eq-cases-2*:
$-x \sqcup -y \leq -x \sqcup -z \Longrightarrow --x \sqcup -y \leq --x \sqcup -z \Longrightarrow -y \leq -z$
by (*metis least-upper-bound sup-less-eq-cases sub-sup-closed*)

lemma *sup-less-eq-cases-3*:
$-y \sqcup -x \leq -z \sqcup -x \Longrightarrow -y \sqcup --x \leq -z \sqcup --x \Longrightarrow -y \leq -z$
by (*simp add*: *sup-less-eq-cases-2 sub-commutative*)

lemma *inf-less-eq-cases*:
$-x \sqcap -y \leq -z \Longrightarrow --x \sqcap -y \leq -z \Longrightarrow -y \leq -z$
by (*simp add*: *shunting sup-less-eq-cases*)

lemma *inf-less-eq-cases-2*:
$-x \sqcap -y \leq -x \sqcap -z \Longrightarrow --x \sqcap -y \leq --x \sqcap -z \Longrightarrow -y \leq -z$
by (*metis greatest-lower-bound inf-closed inf-less-eq-cases*)

lemma *inf-less-eq-cases-3*:
$-y \sqcap -x \leq -z \sqcap -x \Longrightarrow -y \sqcap --x \leq -z \sqcap --x \Longrightarrow -y \leq -z$
by (*simp add*: *inf-commutative inf-less-eq-cases-2*)

lemma *inf-eq-cases*:
$-x \sqcap -y = -x \sqcap -z \Longrightarrow --x \sqcap -y = --x \sqcap -z \Longrightarrow -y = -z$
by (*metis inf-commutative sup-cases*)

lemma *inf-eq-cases-2*:
$-y \sqcap -x = -z \sqcap -x \Longrightarrow -y \sqcap --x = -z \sqcap --x \Longrightarrow -y = -z$
using *inf-commutative inf-eq-cases* **by** *auto*

lemma *wnf-lemma-1*:
$((-x \sqcup -y) \sqcap (--x \sqcup -z)) \sqcup -x = -x \sqcup -y$
proof −
have $\forall u\ v\ w.\ (-u \sqcap (-v \sqcup --w)) \sqcup -w = -u \sqcup -w$
by (*metis inf-right-zero sub-associative sub-sup-closed sup-complement sup-idempotent sup-right-dist-inf*)

thus *?thesis*
  by (*metis* (*no-types*) *sub-associative sub-commutative sub-sup-closed sup-idempotent*)
qed

lemma *wnf-lemma-2*:
  $((-x \sqcup -y) \sqcap (-z \sqcup --y)) \sqcup -y = -x \sqcup -y$
  using *sub-commutative wnf-lemma-1* **by** *fastforce*

lemma *wnf-lemma-3*:
  $((-x \sqcup -z) \sqcap (--x \sqcup -y)) \sqcup --x = --x \sqcup -y$
  by (*metis case-duality case-duality-2 double-negation sub-commutative wnf-lemma-2*)

lemma *wnf-lemma-4*:
  $((-z \sqcup -y) \sqcap (-x \sqcup --y)) \sqcup --y = -x \sqcup --y$
  using *sub-commutative wnf-lemma-3* **by** *auto*

end

class *subset-boolean-algebra'* = *sup* + *uminus* +
  assumes *sub-associative'*: $-x \sqcup (-y \sqcup -z) = (-x \sqcup -y) \sqcup -z$
  assumes *sub-commutative'*: $-x \sqcup -y = -y \sqcup -x$
  assumes *sub-complement'*: $-x = -(--x \sqcup -y) \sqcup -(--x \sqcup --y)$
  assumes *sub-sup-closed'*: $\exists z . -x \sqcup -y = -z$
begin

subclass *subset-boolean-algebra*
proof
  show $\bigwedge x\ y\ z.\ -x \sqcup (-y \sqcup -z) = -x \sqcup -y \sqcup -z$
    by (*simp add*: *sub-associative'*)
  show $\bigwedge x\ y.\ -x \sqcup -y = -y \sqcup -x$
    by (*simp add*: *sub-commutative'*)
  show $\bigwedge x\ y.\ -x = -(--x \sqcup -y) \sqcup -(--x \sqcup --y)$
    by (*simp add*: *sub-complement'*)
  show $\bigwedge x\ y.\ -x \sqcup -y = --(-x \sqcup -y)$
  proof −
    fix *x y*
    have $\forall x\ y.\ -y \sqcup (-(--y \sqcup -x) \sqcup -(---x \sqcup -y)) = -y \sqcup --x$
      by (*metis* (*no-types*) *sub-associative' sub-commutative' sub-complement'*)
    hence $\forall x.\ ---x = -x$
      by (*metis* (*no-types*) *sub-commutative' sub-complement'*)
    thus $-x \sqcup -y = --(-x \sqcup -y)$
      by (*metis sub-sup-closed'*)
  qed
qed

end

We introduce a type for the range of complement and show that it is an instance of *boolean-algebra*.

typedef (**overloaded**) *'a boolean-subset* = { *x*::*'a*::*uminus* . ∃ *y* . *x* = −*y* }
 by *auto*

lemma *simp-boolean-subset*[*simp*]:
 ∃ *y* . *Rep-boolean-subset x* = −*y*
 using *Rep-boolean-subset* **by** *simp*

setup-lifting *type-definition-boolean-subset*

Theorem 8.1

instantiation *boolean-subset* :: (*subset-boolean-algebra*) *huntington*
begin

lift-definition *sup-boolean-subset* :: *'a boolean-subset* ⇒ *'a boolean-subset* ⇒ *'a boolean-subset* **is** *sup*
 using *sub-sup-closed* **by** *auto*

lift-definition *uminus-boolean-subset* :: *'a boolean-subset* ⇒ *'a boolean-subset* **is** *uminus*
 by *auto*

instance
proof
 show $\bigwedge x\ y\ z$::*'a boolean-subset*. $x \sqcup (y \sqcup z) = x \sqcup y \sqcup z$
  apply *transfer*
  using *sub-associative* **by** *blast*
 show $\bigwedge x\ y$::*'a boolean-subset*. $x \sqcup y = y \sqcup x$
  apply *transfer*
  using *sub-commutative* **by** *blast*
 show $\bigwedge x\ y$::*'a boolean-subset*. $x = -(-x \sqcup y) \sqcup -(-x \sqcup -y)$
  apply *transfer*
  using *sub-complement* **by** *blast*
qed

end

Theorem 8.2

instantiation *boolean-subset* :: (*subset-boolean-algebra-extended*) *huntington-extended*
begin

lift-definition *inf-boolean-subset* :: *'a boolean-subset* ⇒ *'a boolean-subset* ⇒ *'a boolean-subset* **is** *inf*
 using *inf-closed* **by** *auto*

lift-definition *minus-boolean-subset* :: *'a boolean-subset* ⇒ *'a boolean-subset* ⇒ *'a boolean-subset* **is** *minus*
 using *sub-minus-def* **by** *auto*

lift-definition *bot-boolean-subset* :: *'a boolean-subset* **is** *bot*
  by (*metis complement-top*)

lift-definition *top-boolean-subset* :: *'a boolean-subset* **is** *top*
  by (*metis complement-bot*)

lift-definition *less-eq-boolean-subset* :: *'a boolean-subset* ⇒ *'a boolean-subset* ⇒ *bool* **is** *less-eq* **.**

lift-definition *less-boolean-subset* :: *'a boolean-subset* ⇒ *'a boolean-subset* ⇒ *bool* **is** *less* **.**

instance
proof
  show *1*: *top* = (*THE x*. ∀ *y*::*'a boolean-subset*. *x* = *y* ⊔ − *y*)
  proof (*rule the-equality*[*symmetric*])
    show ∀ *y*::*'a boolean-subset*. *top* = *y* ⊔ − *y*
      apply *transfer*
      by *auto*
    show ⋀*x*::*'a boolean-subset*. ∀ *y*. *x* = *y* ⊔ − *y* ⟹ *x* = *top*
      apply *transfer*
      by *force*
  qed
  have (*bot*::*'a boolean-subset*) = − *top*
    apply *transfer*
    by *simp*
  thus *bot* = − (*THE x*. ∀ *y*::*'a boolean-subset*. *x* = *y* ⊔ − *y*)
    using *1* **by** *simp*
  show ⋀*x y*::*'a boolean-subset*. *x* ⊓ *y* = − (− *x* ⊔ − *y*)
    apply *transfer*
    using *sub-inf-def* **by** *blast*
  show ⋀*x y*::*'a boolean-subset*. *x* − *y* = − (− *x* ⊔ *y*)
    apply *transfer*
    using *sub-minus-def* **by** *blast*
  show ⋀*x y*::*'a boolean-subset*. (*x* ≤ *y*) = (*x* ⊔ *y* = *y*)
    apply *transfer*
    using *sub-less-eq-def* **by** *blast*
  show ⋀*x y*::*'a boolean-subset*. (*x* < *y*) = (*x* ⊔ *y* = *y* ∧ *y* ⊔ *x* ≠ *x*)
    apply *transfer*
    using *sub-less-def* **by** *blast*
qed

end

5 Subset Boolean algebras with Additional Structure

We now discuss axioms that make the range of a unary operation a Boolean algebra, but add further properties that are common to the intended models. In the intended models, the unary operation can be a complement, a pseudocomplement or the antidomain operation. For simplicity, we mostly call the unary operation 'complement'.

We first look at structures based only on join and complement, and then add axioms for the remaining operations of Boolean algebras. In the intended models, the operation that is meet on the range of the complement can be a meet in the whole algebra or composition.

5.1 Axioms Derived from the New Axiomatisation

The axioms of the first algebra are based on *boolean-algebra-3*.

Definition 9

class *subset-boolean-algebra-1* = *sup* + *uminus* +
  assumes *sba1-associative*: $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$
  assumes *sba1-commutative*: $x \sqcup y = y \sqcup x$
  assumes *sba1-idempotent*[*simp*]: $x \sqcup x = x$
  assumes *sba1-double-complement*[*simp*]: $---x = -x$
  assumes *sba1-bot-unique*: $-(x \sqcup -x) = -(y \sqcup -y)$
  assumes *sba1-export*: $-x \sqcup -(-x \sqcup y) = -x \sqcup -y$
begin

Theorem 11.1

subclass *subset-boolean-algebra*
proof
  show $\bigwedge x\ y\ z.\ -x \sqcup (-y \sqcup -z) = -x \sqcup -y \sqcup -z$
    by (*simp add*: *sba1-associative*)
  show $\bigwedge x\ y.\ -x \sqcup -y = -y \sqcup -x$
    by (*simp add*: *sba1-commutative*)
  show $\bigwedge x\ y.\ -x = -(--x \sqcup -y) \sqcup -(--x \sqcup --y)$
    by (*smt sba1-bot-unique sba1-commutative sba1-double-complement sba1-export sba1-idempotent*)
  thus $\bigwedge x\ y.\ -x \sqcup -y = --(-x \sqcup -y)$
    by (*metis sba1-double-complement sba1-export*)
qed

definition *sba1-bot* $\equiv$ *THE* $x\ .\ \forall z\ .\ x = -(z \sqcup -z)$

lemma *sba1-bot*:
  sba1-bot $= -(z \sqcup -z)$
  using *sba1-bot-def sba1-bot-unique* **by** *auto*

end

Boolean algebra operations based on join and complement

Definition 10

```
class subset-extended-1 = sup + inf + minus + uminus + bot + top + ord +
  assumes ba-bot: bot = (THE x . ∀ z . x = −(z ⊔ −z))
  assumes ba-top: top = −(THE x . ∀ z . x = −(z ⊔ −z))
  assumes ba-inf: −x ⊓ −y = −(−−x ⊔ −−y)
  assumes ba-minus: −x − −y = −(−−x ⊔ −y)
  assumes ba-less-eq: x ≤ y ⟷ x ⊔ y = y
  assumes ba-less: x < y ⟷ x ⊔ y = y ∧ ¬ (y ⊔ x = x)

class subset-extended-2 = subset-extended-1 +
  assumes ba-bot-unique: −(x ⊔ −x) = −(y ⊔ −y)
begin

lemma ba-bot-def:
  bot = −(z ⊔ −z)
  using ba-bot ba-bot-unique by auto

lemma ba-top-def:
  top = −−(z ⊔ −z)
  using ba-bot-def ba-top by simp

end
```

Subset forms Boolean Algebra, extended by Boolean algebra operations

```
class subset-boolean-algebra-1-extended = subset-boolean-algebra-1 +
subset-extended-1
begin

subclass subset-extended-2
proof
  show ⋀x y. − (x ⊔ − x) = − (y ⊔ − y)
    by (simp add: sba1-bot-unique)
qed

subclass semilattice-sup
proof
  show ⋀x y. (x < y) = (x ≤ y ∧ ¬ y ≤ x)
    by (simp add: ba-less ba-less-eq)
  show ⋀x. x ≤ x
    by (simp add: ba-less-eq)
  show ⋀x y z. x ≤ y ⟹ y ≤ z ⟹ x ≤ z
    by (metis sba1-associative ba-less-eq)
  show ⋀x y. x ≤ y ⟹ y ≤ x ⟹ x = y
    by (simp add: sba1-commutative ba-less-eq)
  show ⋀x y. x ≤ x ⊔ y
    by (simp add: sba1-associative ba-less-eq)
  thus ⋀y x. y ≤ x ⊔ y
    by (simp add: sba1-commutative)
  show ⋀y x z. y ≤ x ⟹ z ≤ x ⟹ y ⊔ z ≤ x
    by (metis sba1-associative ba-less-eq)
```

qed

Theorem 11.2

subclass *subset-boolean-algebra-extended*
proof
  show $top = (THE\ x.\ \forall y.\ x = -\ y \sqcup -\ -\ y)$
    by (*smt ba-bot ba-bot-def ba-top sub-sup-closed the-equality*)
  thus $bot = -\ (THE\ x.\ \forall y.\ x = -\ y \sqcup -\ -\ y)$
    using *ba-bot-def ba-top-def* **by** *force*
  show $\bigwedge x\ y.\ -\ x \sqcap -\ y = -\ (-\ -\ x \sqcup -\ -\ y)$
    by (*simp add*: *ba-inf*)
  show $\bigwedge x\ y.\ -\ x\ -\ -\ y = -\ (-\ -\ x \sqcup -\ y)$
    by (*simp add*: *ba-minus*)
  show $\bigwedge x\ y.\ (-\ x \leq -\ y) = (-\ x \sqcup -\ y = -\ y)$
    using *le-iff-sup* **by** *auto*
  show $\bigwedge x\ y.\ (-\ x < -\ y) = (-\ x \sqcup -\ y = -\ y \wedge -\ y \sqcup -\ x \neq -\ x)$
    by (*simp add*: *ba-less*)
qed

end

5.2 Stronger Assumptions based on Join and Complement

We add further axioms covering properties common to the antidomain and (pseudo)complement instances.

Definition 12

class *subset-boolean-algebra-2* = *sup* + *uminus* +
  assumes *sba2-associative*: $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$
  assumes *sba2-commutative*: $x \sqcup y = y \sqcup x$
  assumes *sba2-idempotent*[*simp*]: $x \sqcup x = x$
  assumes *sba2-bot-unit*: $x \sqcup -(y \sqcup -y) = x$
  assumes *sba2-sub-sup-demorgan*: $-(x \sqcup y) = -(--x \sqcup --y)$
  assumes *sba2-export*: $-x \sqcup -(-x \sqcup y) = -x \sqcup -y$
begin

Theorem 13.1

subclass *subset-boolean-algebra-1*
proof
  show $\bigwedge x\ y\ z.\ x \sqcup (y \sqcup z) = x \sqcup y \sqcup z$
    by (*simp add*: *sba2-associative*)
  show $\bigwedge x\ y.\ x \sqcup y = y \sqcup x$
    by (*simp add*: *sba2-commutative*)
  show $\bigwedge x.\ x \sqcup x = x$
    by *simp*
  show $\bigwedge x.\ -\ -\ -\ x = -\ x$
    by (*metis sba2-idempotent sba2-sub-sup-demorgan*)
  show $\bigwedge x\ y.\ -\ (x \sqcup -\ x) = -\ (y \sqcup -\ y)$
    by (*metis sba2-bot-unit sba2-commutative*)

show $\bigwedge x\ y.\ -\ x \sqcup -\ (-\ x \sqcup y) = -\ x \sqcup -\ y$
by (*simp add*: *sba2-export*)
qed

Theorem 13.2

lemma *double-complement-dist-sup*:
$--(x \sqcup y) = --x \sqcup --y$
by (*metis sba2-commutative sba2-export sba2-idempotent sba2-sub-sup-demorgan*)

lemma *maddux-3-3*[*simp*]:
$-(x \sqcup y) \sqcup -(x \sqcup -y) = -x$
by (*metis double-complement-dist-sup sba1-double-complement sba2-commutative sub-complement*)

lemma *huntington-3-pp*[*simp*]:
$-(-x \sqcup -y) \sqcup -(-x \sqcup y) = --x$
using *sba2-commutative maddux-3-3* **by** *fastforce*

end

class *subset-boolean-algebra-2-extended* = *subset-boolean-algebra-2* + *subset-extended-1*
begin

subclass *subset-boolean-algebra-1-extended* **..**

subclass *bounded-semilattice-sup-bot*
proof
show $\bigwedge x.\ bot \leq x$
using *sba2-bot-unit ba-bot-def sup-right-divisibility* **by** *auto*
qed

Theorem 13.3

lemma *complement-antitone*:
$x \leq y \Longrightarrow -y \leq -x$
by (*metis le-iff-sup maddux-3-3 sba2-export sup-monoid.add-commute*)

lemma *double-complement-isotone*:
$x \leq y \Longrightarrow --x \leq --y$
by (*simp add*: *complement-antitone*)

lemma *sup-demorgan*:
$-(x \sqcup y) = -x \sqcap -y$
using *sba2-sub-sup-demorgan ba-inf* **by** *auto*

end

5.3 Axioms for Meet

We add further axioms of *inf* covering properties common to the antidomain and pseudocomplement instances. We omit the left distributivity rule and the right zero rule as they do not hold in some models. In particular, the operation *inf* does not have to be commutative.

Definition 14

class *subset-boolean-algebra-3-extended* = *subset-boolean-algebra-2-extended* +
  assumes *sba3-inf-associative*: $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$
  assumes *sba3-inf-right-dist-sup*: $(x \sqcup y) \sqcap z = (x \sqcap z) \sqcup (y \sqcap z)$
  assumes *sba3-inf-complement-bot*: $-x \sqcap x = bot$
  assumes *sba3-inf-left-unit*[*simp*]: $top \sqcap x = x$
  assumes *sba3-complement-inf-double-complement*: $-(x \sqcap --y) = -(x \sqcap y)$
begin

Theorem 15

lemma *inf-left-zero*:
  $bot \sqcap x = bot$
  by (*metis inf-right-unit sba3-inf-associative sba3-inf-complement-bot*)

lemma *inf-double-complement-export*:
  $--(--x \sqcap y) = --x \sqcap --y$
  by (*metis inf-closed sba3-complement-inf-double-complement*)

lemma *inf-left-isotone*:
  $x \leq y \Longrightarrow x \sqcap z \leq y \sqcap z$
  using *sba3-inf-right-dist-sup sup-right-divisibility* **by** *auto*

lemma *inf-complement-export*:
  $--(-x \sqcap y) = -x \sqcap --y$
  by (*metis inf-double-complement-export sba1-double-complement*)

lemma *double-complement-above*:
  $--x \sqcap x = x$
  by (*metis sup-monoid.add-0-right complement-bot inf-demorgan sba1-double-complement sba3-inf-complement-bot sba3-inf-right-dist-sup sba3-inf-left-unit*)

lemma $x \leq y \Longrightarrow z \sqcap x \leq z \sqcap y$ **nitpick** [*expect=genuine*] **oops**
lemma $x \sqcap top = x$ **nitpick** [*expect=genuine*] **oops**
lemma $x \sqcap y = y \sqcap x$ **nitpick** [*expect=genuine*] **oops**

end

5.4 Stronger Assumptions for Meet

The following axioms also hold in both models, but follow from the axioms of *subset-boolean-algebra-5-operations*.

Definition 16

class *subset-boolean-algebra-4-extended* = *subset-boolean-algebra-3-extended* +
assumes *sba4-inf-right-unit*[*simp*]: $x \sqcap top = x$
assumes *inf-right-isotone*: $x \le y \Longrightarrow z \sqcap x \le z \sqcap y$
begin

lemma $x \sqcup top = top$ **nitpick** [*expect=genuine*] **oops**
lemma $x \sqcap bot = bot$ **nitpick** [*expect=genuine*] **oops**
lemma $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$ **nitpick** [*expect=genuine*] **oops**
lemma $(x \sqcap y = bot) = (x \le - y)$ **nitpick** [*expect=genuine*] **oops**

end

6 Boolean Algebras in Stone Algebras

We specialise *inf* to meet and complement to pseudocomplement. This puts Stone algebras into the picture; for these it is well known that regular elements form a Boolean subalgebra [12].

Definition 17

class *subset-boolean-algebra-5-extended* = *subset-boolean-algebra-3-extended* +
assumes *sba5-inf-commutative*: $x \sqcap y = y \sqcap x$
assumes *sba5-inf-absorb*: $x \sqcap (x \sqcup y) = x$
begin

subclass *distrib-lattice-bot*
proof
show $\bigwedge x\ y.\ x \sqcap y \le x$
by (*metis sba5-inf-commutative sba3-inf-right-dist-sup sba5-inf-absorb sup-right-divisibility*)
show $\bigwedge x\ y.\ x \sqcap y \le y$
by (*metis inf-left-isotone sba5-inf-absorb sba5-inf-commutative sup-ge2*)
show $\bigwedge x\ y\ z.\ x \le y \Longrightarrow x \le z \Longrightarrow x \le y \sqcap z$
by (*metis inf-left-isotone sba5-inf-absorb sup.orderE sup-monoid.add-commute*)
show $\bigwedge x\ y\ z.\ x \sqcup y \sqcap z = (x \sqcup y) \sqcap (x \sqcup z)$
by (*metis sba3-inf-right-dist-sup sba5-inf-absorb sba5-inf-commutative sup-assoc*)
qed

lemma *inf-demorgan-2*:
$-(x \sqcap y) = -x \sqcup -y$
using *sba3-complement-inf-double-complement sba5-inf-commutative sub-sup-closed sub-sup-demorgan* **by** *auto*

lemma *inf-export*:
$x \sqcap -(x \sqcap y) = x \sqcap -y$
using *inf-demorgan-2 sba3-inf-complement-bot sba3-inf-right-dist-sup sba5-inf-commutative* **by** *auto*

lemma *complement-inf* [*simp*]:
  $x \sqcap -x = bot$
  using *sba3-inf-complement-bot sba5-inf-commutative* **by** *auto*

Theorem 18.2

subclass *stone-algebra*
proof
  show $\bigwedge x.\ x \leq top$
    by (*simp add*: *inf.absorb-iff2*)
  show $\bigwedge x\ y.\ (x \sqcap y = bot) = (x \leq -\ y)$
    by (*metis* (*full-types*) *complement-bot complement-inf inf.cobounded1 inf.order-iff inf-export sba3-complement-inf-double-complement sba3-inf-left-unit*)
  show $\bigwedge x.\ -\ x \sqcup -\ -\ x = top$
    by *simp*
qed

Theorem 18.1

subclass *subset-boolean-algebra-4-extended*
proof
  show $\bigwedge x.\ x \sqcap top = x$
    by *simp*
  show $\bigwedge x\ y\ z.\ x \leq y \Longrightarrow z \sqcap x \leq z \sqcap y$
    using *inf.sup-right-isotone* **by** *blast*
qed

end

context *stone-algebra-extended*
begin

Theorem 18.3

subclass *subset-boolean-algebra-5-extended*
proof
  show $\bigwedge x\ y\ z.\ x \sqcup (y \sqcup z) = x \sqcup y \sqcup z$
    using *sup-assoc* **by** *auto*
  show $\bigwedge x\ y.\ x \sqcup y = y \sqcup x$
    by (*simp add*: *sup-commute*)
  show $\bigwedge x.\ x \sqcup x = x$
    by *simp*
  show $\bigwedge x\ y.\ x \sqcup -\ (y \sqcup -\ y) = x$
    by *simp*
  show $\bigwedge x\ y.\ -\ (x \sqcup y) = -\ (-\ -\ x \sqcup -\ -\ y)$
    by *auto*
  show $\bigwedge x\ y.\ -\ x \sqcup -\ (-\ x \sqcup y) = -\ x \sqcup -\ y$
    by (*metis maddux-3-21-pp p-dist-sup regular-closed-p*)
  show $bot = (THE\ x.\ \forall z.\ x = -\ (z \sqcup -\ z))$
    by *simp*
  thus $top = -\ (THE\ x.\ \forall z.\ x = -\ (z \sqcup -\ z))$

using *p-bot* **by** *blast*
show $\bigwedge x\ y.\ -\ x \sqcap -\ y = -\ (-\ -\ x \sqcup -\ -\ y)$
by *simp*
show $\bigwedge x\ y.\ -\ x -\ -\ y = -\ (-\ -\ x \sqcup -\ y)$
by *auto*
show $\bigwedge x\ y.\ (x \leq y) = (x \sqcup y = y)$
by (*simp add*: *le-iff-sup*)
thus $\bigwedge x\ y.\ (x < y) = (x \sqcup y = y \wedge y \sqcup x \neq x)$
by (*simp add*: *less-le-not-le*)
show $\bigwedge x\ y\ z.\ x \sqcap (y \sqcap z) = x \sqcap y \sqcap z$
by (*simp add*: *inf.sup-monoid.add-assoc*)
show $\bigwedge x\ y\ z.\ (x \sqcup y) \sqcap z = x \sqcap z \sqcup y \sqcap z$
by (*simp add*: *inf-sup-distrib2*)
show $\bigwedge x.\ -\ x \sqcap x = bot$
by *simp*
show $\bigwedge x.\ top \sqcap x = x$
by *simp*
show $\bigwedge x\ y.\ -\ (x \sqcap -\ -\ y) = -\ (x \sqcap y)$
by *simp*
show $\bigwedge x\ y.\ x \sqcap y = y \sqcap x$
by (*simp add*: *inf-commute*)
show $\bigwedge x\ y.\ x \sqcap (x \sqcup y) = x$
by *simp*
qed

end

7 Domain Semirings

The following development of tests in IL-semirings, prepredomain semirings, predomain semirings and domain semirings is mostly based on [23]; see also [4]. See [5] for domain axioms in idempotent semirings. See [3, 19] for domain axioms in semigroups and monoids. Some variants have been implemented in [11].

7.1 Idempotent Left Semirings

Definition 19

class *il-semiring* = *sup* + *inf* + *bot* + *top* + *ord* +
assumes *il-associative*: $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$
assumes *il-commutative*: $x \sqcup y = y \sqcup x$
assumes *il-idempotent*[*simp*]: $x \sqcup x = x$
assumes *il-bot-unit*: $x \sqcup bot = x$
assumes *il-inf-associative*: $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$
assumes *il-inf-right-dist-sup*: $(x \sqcup y) \sqcap z = (x \sqcap z) \sqcup (y \sqcap z)$
assumes *il-inf-left-unit*[*simp*]: $top \sqcap x = x$
assumes *il-inf-right-unit*[*simp*]: $x \sqcap top = x$

assumes *il-sub-inf-left-zero*[*simp*]: *bot* ⊓ *x* = *bot*
assumes *il-sub-inf-right-isotone*: *x* ≤ *y* ⟹ *z* ⊓ *x* ≤ *z* ⊓ *y*
assumes *il-less-eq*: *x* ≤ *y* ⟷ *x* ⊔ *y* = *y*
assumes *il-less-def*: *x* < *y* ⟷ *x* ≤ *y* ∧ ¬(*y* ≤ *x*)
begin

lemma *il-unit-bot*: *bot* ⊔ *x* = *x*
using *il-bot-unit il-commutative* **by** *fastforce*

subclass *order*
proof
show ⋀*x y*. (*x* < *y*) = (*x* ≤ *y* ∧ ¬ *y* ≤ *x*)
by (*simp add*: *il-less-def*)
show ⋀*x*. *x* ≤ *x*
by (*simp add*: *il-less-eq*)
show ⋀*x y z*. *x* ≤ *y* ⟹ *y* ≤ *z* ⟹ *x* ≤ *z*
by (*metis il-associative il-less-eq*)
show ⋀*x y*. *x* ≤ *y* ⟹ *y* ≤ *x* ⟹ *x* = *y*
by (*simp add*: *il-commutative il-less-eq*)
qed

lemma *il-sub-inf-right-isotone-var*:
(*x* ⊓ *y*) ⊔ (*x* ⊓ *z*) ≤ *x* ⊓ (*y* ⊔ *z*)
by (*smt il-associative il-commutative il-idempotent il-less-eq il-sub-inf-right-isotone*)

lemma *il-sub-inf-left-isotone*:
x ≤ *y* ⟹ *x* ⊓ *z* ≤ *y* ⊓ *z*
by (*metis il-inf-right-dist-sup il-less-eq*)

lemma *il-sub-inf-left-isotone-var*:
(*y* ⊓ *x*) ⊔ (*z* ⊓ *x*) ≤ (*y* ⊔ *z*) ⊓ *x*
by (*simp add*: *il-inf-right-dist-sup*)

lemma *sup-left-isotone*:
x ≤ *y* ⟹ *x* ⊔ *z* ≤ *y* ⊔ *z*
by (*smt il-associative il-commutative il-idempotent il-less-eq*)

lemma *sup-right-isotone*:
x ≤ *y* ⟹ *z* ⊔ *x* ≤ *z* ⊔ *y*
by (*simp add*: *il-commutative sup-left-isotone*)

lemma *bot-least*:
bot ≤ *x*
by (*simp add*: *il-less-eq il-unit-bot*)

lemma *less-eq-bot*:
x ≤ *bot* ⟷ *x* = *bot*
by (*simp add*: *il-bot-unit il-less-eq*)

abbreviation *are-complementary* :: $'a \Rightarrow 'a \Rightarrow$ *bool*
  where *are-complementary* $x\ y \equiv x \sqcup y = top \wedge x \sqcap y = bot \wedge y \sqcap x = bot$

abbreviation *test* :: $'a \Rightarrow$ *bool*
  where *test* $x \equiv \exists\, y$. *are-complementary* $x\ y$

definition *tests* :: $'a$ *set*
  where *tests* $= \{\ x\ .\ test\ x\ \}$

lemma *bot-test*:
  test bot
  by (*simp add*: *il-unit-bot*)

lemma *top-test*:
  test top
  by (*simp add*: *il-bot-unit*)

lemma *test-sub-identity*:
  test $x \Longrightarrow x \leq top$
  using *il-associative il-less-eq* **by** *auto*

lemma *neg-unique*:
  are-complementary $x\ y \Longrightarrow$ *are-complementary* $x\ z \Longrightarrow y = z$
  by (*metis antisym il-inf-left-unit il-inf-right-dist-sup il-inf-right-unit il-sub-inf-right-isotone-var*)

definition *neg* :: $'a \Rightarrow 'a$ (!)
  where $!x \equiv$ *THE* y . *are-complementary* $x\ y$

lemma *neg-char*:
  assumes *test* x
    shows *are-complementary* x $(!x)$
proof (*unfold neg-def*)
  from *assms* **obtain** y **where** *1*: *are-complementary* $x\ y$
    by *auto*
  show *are-complementary* x (*THE* y. *are-complementary* $x\ y$)
  proof (*rule theI*)
    show *are-complementary* $x\ y$
      using *1* **by** *simp*
    show $\bigwedge z$. *are-complementary* $x\ z \Longrightarrow z = y$
      using *1 neg-unique* **by** *blast*
  qed
qed

lemma *are-complementary-symmetric*:
  are-complementary $x\ y \longleftrightarrow$ *are-complementary* $y\ x$
  using *il-commutative* **by** *auto*

lemma *neg-test*:
test $x \Longrightarrow$ *test* $(!x)$
using *are-complementary-symmetric neg-char* **by** *blast*

lemma *are-complementary-test*:
test $x \Longrightarrow$ *are-complementary* x $y \Longrightarrow$ *test* y
using *il-commutative* **by** *auto*

lemma *neg-involutive*:
test $x \Longrightarrow !(!x) = x$
using *are-complementary-symmetric neg-char neg-unique* **by** *blast*

lemma *test-inf-left-below*:
test $x \Longrightarrow x \sqcap y \leq y$
by (*metis il-associative il-idempotent il-inf-left-unit il-inf-right-dist-sup il-less-eq*)

lemma *test-inf-right-below*:
test $x \Longrightarrow y \sqcap x \leq y$
by (*metis il-inf-right-unit il-sub-inf-right-isotone test-sub-identity*)

lemma *neg-bot*:
$!bot = top$
using *il-unit-bot neg-char* **by** *fastforce*

lemma *neg-top*:
$!top = bot$
using *bot-test neg-bot neg-involutive* **by** *fastforce*

lemma *test-inf-idempotent*:
test $x \Longrightarrow x \sqcap x = x$
by (*metis il-bot-unit il-inf-left-unit il-inf-right-dist-sup*)

lemma *test-inf-semicommutative*:
assumes *test* x
and *test* y
shows $x \sqcap y \leq y \sqcap x$
proof −
have $x \sqcap y = (y \sqcap x \sqcap y) \sqcup (!y \sqcap x \sqcap y)$
by (*metis assms(2) il-inf-left-unit il-inf-right-dist-sup neg-char*)
also have ... $\leq (y \sqcap x \sqcap y) \sqcup (!y \sqcap y)$
proof −
obtain z **where** *are-complementary* y z
using *assms(2)* **by** *blast*
hence $y \sqcap (x \sqcap y) \sqcup !y \sqcap (x \sqcap y) \leq y \sqcap (x \sqcap y)$
by (*metis assms(1) calculation il-sub-inf-left-isotone il-bot-unit il-idempotent il-inf-associative il-less-eq neg-char test-inf-right-below*)
thus *?thesis*
by (*simp add*: *il-associative il-inf-associative il-less-eq*)

qed
also have ... $\leq (y \sqcap x) \sqcup (!y \sqcap y)$
by (*metis assms*(*2*) *il-bot-unit il-inf-right-unit il-sub-inf-right-isotone neg-char test-sub-identity*)
also have ... $= y \sqcap x$
by (*simp add*: *assms*(*2*) *il-bot-unit neg-char*)
finally show *?thesis*
.
qed

lemma *test-inf-commutative*:
test $x \Longrightarrow$ *test* $y \Longrightarrow x \sqcap y = y \sqcap x$
by (*simp add*: *antisym test-inf-semicommutative*)

lemma *test-inf-bot*:
test $x \Longrightarrow x \sqcap$ *bot* = *bot*
using *il-inf-associative test-inf-idempotent* **by** *fastforce*

lemma *test-absorb-1*:
test $x \Longrightarrow$ *test* $y \Longrightarrow x \sqcup (x \sqcap y) = x$
using *il-commutative il-less-eq test-inf-right-below* **by** *auto*

lemma *test-absorb-2*:
test $x \Longrightarrow$ *test* $y \Longrightarrow x \sqcup (y \sqcap x) = x$
by (*metis test-absorb-1 test-inf-commutative*)

lemma *test-absorb-3*:
test $x \Longrightarrow$ *test* $y \Longrightarrow x \sqcap (x \sqcup y) = x$
apply (*rule antisym*)
apply (*metis il-associative il-inf-right-unit il-less-eq il-sub-inf-right-isotone test-sub-identity*)
by (*metis il-sub-inf-right-isotone-var test-absorb-1 test-inf-idempotent*)

lemma *test-absorb-4*:
test $x \Longrightarrow$ *test* $y \Longrightarrow (x \sqcup y) \sqcap x = x$
by (*smt il-inf-right-dist-sup test-inf-idempotent il-commutative il-less-eq test-inf-left-below*)

lemma *test-import-1*:
assumes *test x*
and *test y*
shows $x \sqcup (!x \sqcap y) = x \sqcup y$
proof −
have $x \sqcup (!x \sqcap y) = x \sqcup ((y \sqcup !y) \sqcap x) \sqcup (!x \sqcap y)$
by (*simp add*: *assms*(*2*) *neg-char*)
also have ... $= x \sqcup (!y \sqcap x) \sqcup (x \sqcap y) \sqcup (!x \sqcap y)$
by (*smt assms il-associative il-commutative il-inf-right-dist-sup test-inf-commutative*)
also have ... $= x \sqcup ((x \sqcup !x) \sqcap y)$

by (*smt calculation il-associative il-commutative il-idempotent il-inf-right-dist-sup*)
also have ... = $x \sqcup y$
by (*simp add*: *assms*(*1*) *neg-char*)
finally show *?thesis*
.
qed

lemma *test-import-2*:
assumes *test x*
and *test y*
shows $x \sqcup (y \sqcap !x) = x \sqcup y$
proof −
obtain z **where** *1*: *are-complementary y z*
using *assms*(*2*) **by** *moura*
obtain w **where** *2*: *are-complementary x w*
using *assms*(*1*) **by** *auto*
hence $x \sqcap !x = bot$
using *neg-char* **by** *blast*
hence $!x \sqcap y = y \sqcap !x$
using *1 2* **by** (*metis il-commutative neg-char test-inf-commutative*)
thus *?thesis*
using *1 2* **by** (*metis test-import-1*)
qed

lemma *test-import-3*:
assumes *test x*
shows $(!x \sqcup y) \sqcap x = y \sqcap x$
by (*simp add*: *assms*(*1*) *il-inf-right-dist-sup il-unit-bot neg-char*)

lemma *test-import-4*:
assumes *test x*
and *test y*
shows $(!x \sqcup y) \sqcap x = x \sqcap y$
by (*metis assms test-import-3 test-inf-commutative*)

lemma *test-inf*:
test x $\Longrightarrow$ *test y* $\Longrightarrow$ *test z* $\Longrightarrow z \leq x \sqcap y \longleftrightarrow z \leq x \wedge z \leq y$
apply (*rule iffI*)
using *dual-order.trans test-inf-left-below test-inf-right-below* **apply** *blast*
by (*smt il-less-eq il-sub-inf-right-isotone test-absorb-4*)

lemma *test-shunting*:
assumes *test x*
and *test y*
shows $x \sqcap y \leq z \longleftrightarrow x \leq !y \sqcup z$
proof
assume *1*: $x \sqcap y \leq z$
have $x = (!y \sqcap x) \sqcup (y \sqcap x)$

by (*metis assms*(*2*) *il-commutative il-inf-left-unit il-inf-right-dist-sup neg-char*)
also have ... $\leq$!$y \sqcup (y \sqcap x)$
by (*simp add*: *assms*(*1*) *sup-left-isotone test-inf-right-below*)
also have ... $\leq$!$y \sqcup z$
using *1* **by** (*simp add*: *assms sup-right-isotone test-inf-commutative*)
finally show $x \leq$!$y \sqcup z$
.
next
assume $x \leq$!$y \sqcup z$
hence $x \sqcap y \leq$ (!$y \sqcup z) \sqcap y$
using *il-sub-inf-left-isotone* **by** *blast*
also have ... $= z \sqcap y$
by (*simp add*: *assms*(*2*) *test-import-3*)
also have ... $\leq z$
by (*simp add*: *assms*(*2*) *test-inf-right-below*)
finally show $x \sqcap y \leq z$
.
qed

lemma *test-shunting-bot*:
assumes *test x*
and *test y*
shows $x \leq y \longleftrightarrow x \sqcap$!$y \leq$ *bot*
by (*simp add*: *assms il-bot-unit neg-involutive neg-test test-shunting*)

lemma *test-shunting-bot-eq*:
assumes *test x*
and *test y*
shows $x \leq y \longleftrightarrow x \sqcap$!$y =$ *bot*
by (*simp add*: *assms test-shunting-bot less-eq-bot*)

lemma *neg-antitone*:
assumes *test x*
and *test y*
and $x \leq y$
shows !$y \leq$!x
proof −
have *1*: $x \sqcap$!$y =$ *bot*
using *assms test-shunting-bot-eq* **by** *blast*
have *2*: $x \sqcup$!$x =$ *top*
by (*simp add*: *assms*(*1*) *neg-char*)
have *are-complementary y* (!y)
by (*simp add*: *assms*(*2*) *neg-char*)
thus *?thesis*
using *1 2* **by** (*metis il-unit-bot il-commutative il-inf-left-unit*
il-inf-right-dist-sup il-inf-right-unit il-sub-inf-right-isotone test-sub-identity)
qed

lemma *test-sup-neg-1*:

assumes *test x*
and *test y*
shows $(x \sqcup y) \sqcup (!x \sqcap !y) = top$
proof –
have $x \sqcup !x = top$
by (*simp add*: *assms*(*1*) *neg-char*)
hence $x \sqcup (y \sqcup !x) = top$
by (*metis assms*(*2*) *il-associative il-commutative il-idempotent*)
hence $x \sqcup (y \sqcup !x \sqcap !y) = top$
by (*simp add*: *assms neg-test test-import-2*)
thus *?thesis*
by (*simp add*: *il-associative*)
qed

lemma *test-sup-neg-2*:
assumes *test x*
and *test y*
shows $(x \sqcup y) \sqcap (!x \sqcap !y) = bot$
proof –
have *1*: *are-complementary y* (!*y*)
by (*simp add*: *assms*(*2*) *neg-char*)
obtain *z* **where** *2*: *are-complementary x z*
using *assms*(*1*) **by** *auto*
hence $!x = z$
using *neg-char neg-unique* **by** *blast*
thus *?thesis*
using *1 2* **by** (*metis are-complementary-symmetric il-inf-associative neg-involutive test-import-3 test-inf-bot test-inf-commutative*)
qed

lemma *de-morgan-1*:
assumes *test x*
and *test y*
and *test* $(x \sqcap y)$
shows $!(x \sqcap y) = !x \sqcup !y$
proof (*rule antisym*)
have *1*: *test* $(!(x \sqcap y))$
by (*simp add*: *assms neg-test*)
have $x \le (x \sqcap y) \sqcup !y$
by (*metis* (*full-types*) *assms il-commutative neg-char test-shunting test-shunting-bot-eq*)
hence $x \sqcap !(x \sqcap y) \le !y$
using *1* **by** (*simp add*: *assms*(*1*,*3*) *neg-involutive test-shunting*)
hence $!(x \sqcap y) \sqcap x \le !y$
using *1* **by** (*metis assms*(*1*) *test-inf-commutative*)
thus $!(x \sqcap y) \le !x \sqcup !y$
using *1 assms*(*1*) *test-shunting* **by** *blast*
have *2*: $!x \le !(x \sqcap y)$
by (*simp add*: *assms neg-antitone test-inf-right-below*)

have $!y \leq !(x \sqcap y)$
by (*simp add*: *assms neg-antitone test-inf-left-below*)
thus $!x \sqcup !y \leq !(x \sqcap y)$
using *2* **by** (*metis il-associative il-less-eq*)
qed

lemma *de-morgan-2*:
assumes *test x*
and *test y*
and *test* $(x \sqcup y)$
shows $!(x \sqcup y) = !x \sqcap !y$
proof (*rule antisym*)
have *1*: $!(x \sqcup y) \leq !x$
by (*metis assms il-inf-left-unit il-sub-inf-left-isotone neg-antitone test-absorb-3 test-sub-identity*)
have $!(x \sqcup y) \leq !y$
by (*metis assms il-commutative il-inf-left-unit il-sub-inf-left-isotone neg-antitone test-absorb-3 test-sub-identity*)
thus $!(x \sqcup y) \leq !x \sqcap !y$
using *1* **by** (*simp add*: *assms neg-test test-inf*)
have $top \leq x \sqcup y \sqcup !(x \sqcup y)$
by (*simp add*: *assms*(*3*) *neg-char*)
hence $top \sqcap !x \leq y \sqcup !(x \sqcup y)$
by (*smt assms*(*1*) *assms*(*3*) *il-commutative il-inf-right-dist-sup il-inf-right-unit il-sub-inf-right-isotone il-unit-bot neg-char test-sub-identity*)
thus $!x \sqcap !y \leq !(x \sqcup y)$
by (*simp add*: *assms*(*1*) *assms*(*2*) *neg-involutive neg-test test-shunting*)
qed

lemma *test-inf-closed-sup-complement*:
assumes *test x*
and *test y*
and $\forall u\ v$. *test u* $\wedge$ *test v* $\longrightarrow$ *test* $(u \sqcap v)$
shows $!x \sqcap !y \sqcap (x \sqcup y) = bot$
proof −
have *1*: $!(!x \sqcap !y) = x \sqcup y$
by (*simp add*: *assms de-morgan-1 neg-involutive neg-test*)
have *test* $(!(!x \sqcap !y))$
by (*metis assms neg-test*)
thus *?thesis*
using *1* **by** (*metis assms*(*1*,*2*) *de-morgan-2 neg-char*)
qed

lemma *test-sup-complement-sup-closed*:
assumes *test x*
and *test y*
and $\forall u\ v$. *test u* $\wedge$ *test v* $\longrightarrow$ $!u \sqcap !v \sqcap (u \sqcup v) = bot$
shows *test* $(x \sqcup y)$
by (*meson assms test-sup-neg-1 test-sup-neg-2*)

lemma *test-inf-closed-sup-closed*:
  assumes *test x*
    and *test y*
    and $\forall$ *u v* . *test u* $\wedge$ *test v* $\longrightarrow$ *test* $(u \sqcap v)$
  shows *test* $(x \sqcup y)$
  using *assms test-inf-closed-sup-complement test-sup-complement-sup-closed* **by** *simp*

end

7.2 Prepredomain Semirings

class *dom* =
  fixes $d :: 'a \Rightarrow 'a$

class *ppd-semiring* = *il-semiring* + *dom* +
  assumes *d-closed*: *test* $(d\ x)$
  assumes *d1*: $x \leq d\ x \sqcap x$
begin

lemma *d-sub-identity*:
  $d\ x \leq top$
  using *d-closed test-sub-identity* **by** *blast*

lemma *d1-eq*:
  $x = d\ x \sqcap x$
proof −
  have $x = (d\ x \sqcup top) \sqcap x$
    using *d-sub-identity il-less-eq* **by** *auto*
  thus *?thesis*
    using *d1 il-commutative il-inf-right-dist-sup il-less-eq* **by** *force*
qed

lemma *d-increasing-sub-identity*:
  $x \leq top \Longrightarrow x \leq d\ x$
  by (*metis d1-eq il-inf-right-unit il-sub-inf-right-isotone*)

lemma *d-top*:
  $d\ top = top$
  by (*simp add*: *d-increasing-sub-identity d-sub-identity dual-order.antisym*)

lemma *d-bot-only*:
  $d\ x = bot \Longrightarrow x = bot$
  by (*metis d1-eq il-sub-inf-left-zero*)

lemma *d-strict*: $d\ bot \leq bot$ **nitpick** [*expect=genuine*] **oops**
lemma *d-isotone-var*: $d\ x \leq d\ (x \sqcup y)$ **nitpick** [*expect=genuine*] **oops**
lemma *d-fully-strict*: $d\ x = bot \longleftrightarrow x = bot$ **nitpick** [*expect=genuine*] **oops**

lemma *test-d-fixpoint*: *test* $x \Longrightarrow d\ x = x$ **nitpick** [*expect*=*genuine*] **oops**

end

7.3 Predomain Semirings

class *pd-semiring* = *ppd-semiring* +
  assumes *d2*: *test* $p \Longrightarrow d\ (p \sqcap x) \leq p$
begin

lemma *d-strict*:
  $d\ bot \leq bot$
  using *bot-test d2* **by** *fastforce*

lemma *d-strict-eq*:
  $d\ bot = bot$
  using *d-strict il-bot-unit il-less-eq* **by** *auto*

lemma *test-d-fixpoint*:
  test $x \Longrightarrow d\ x = x$
  by (*metis antisym d1-eq d2 test-inf-idempotent test-inf-right-below*)

lemma *d-surjective*:
  test $x \Longrightarrow \exists\, y\ .\ d\ y = x$
  using *test-d-fixpoint* **by** *blast*

lemma *test-d-fixpoint-iff*:
  test $x \longleftrightarrow d\ x = x$
  by (*metis d-closed test-d-fixpoint*)

lemma *d-surjective-iff*:
  test $x \longleftrightarrow (\exists\, y\ .\ d\ y = x)$
  using *d-surjective d-closed* **by** *blast*

lemma *tests-d-range*:
  tests = *range d*
  using *tests-def image-def d-surjective-iff* **by** *auto*

lemma *llp*:
  assumes *test y*
    shows $d\ x \leq y \longleftrightarrow x \leq y \sqcap x$
  by (*metis assms d1-eq d2 eq-iff il-sub-inf-left-isotone test-inf-left-below*)

lemma *gla*:
  assumes *test y*
    shows $y \leq\ !(d\ x) \longleftrightarrow y \sqcap x \leq bot$
proof −
  obtain *ad* **where** *1*: $\forall\, x.$ *are-complementary* $(d\ x)\ (ad\ x)$
    using *d-closed* **by** *moura*

hence *2*: ∀ *x y*. *d* (*d y* ⊓ *x*) ≤ *d y*
using *d2* **by** *blast*
have *3*: ∀ *x*. *ad x* ⊓ *x* = *bot*
using *1* **by** (*metis d1-eq il-inf-associative il-sub-inf-left-zero*)
have *4*: ∀ *x y*. *d y* ⊓ *x* ⊔ *ad y* ⊓ *x* = *top* ⊓ *x*
using *1* **by** (*metis il-inf-right-dist-sup*)
have *5*: ∀ *x y z*. *z* ⊓ *y* ≤ *x* ⊓ *y* ∨ (*z* ⊔ *x*) ⊓ *y* ≠ *x* ⊓ *y*
by (*simp add*: *il-inf-right-dist-sup il-less-eq*)
have *6*: ∀ *x*. !(*d x*) = *ad x*
using *1 neg-char neg-unique* **by** *blast*
have *7*: ∀ *x*. *top* ⊓ *x* = *x*
by *auto*
hence ∀ *x*. *y* ⊓ *x* ⊔ !*y* ⊓ *x* = *x*
by (*metis assms il-inf-right-dist-sup neg-char*)
thus *?thesis*
using *1 2 3 4 5 6 7* **by** (*metis assms d1-eq il-commutative il-less-eq test-d-fixpoint*)
qed

lemma *gla-var*:
test y ⟹ *y* ⊓ *d x* ≤ *bot* ⟷ *y* ⊓ *x* ≤ *bot*
using *gla d-closed il-bot-unit test-shunting* **by** *auto*

lemma *llp-var*:
assumes *test y*
shows *y* ≤ !(*d x*) ⟷ *x* ≤ !*y* ⊓ *x*
apply (*rule iffI*)
apply (*metis* (*no-types*, *hide-lams*) *assms gla Least-equality il-inf-left-unit il-inf-right-dist-sup il-less-eq il-unit-bot order.refl neg-char*)
by (*metis assms gla gla-var llp il-commutative il-sub-inf-right-isotone neg-char*)

lemma *d-idempotent*:
d (*d x*) = *d x*
using *d-closed test-d-fixpoint-iff* **by** *auto*

lemma *d-neg*:
test x ⟹ *d* (!*x*) = !*x*
using *il-commutative neg-char test-d-fixpoint-iff* **by** *fastforce*

lemma *d-fully-strict*:
d x = *bot* ⟷ *x* = *bot*
using *d-strict-eq d-bot-only* **by** *blast*

lemma *d-ad-comp*:
!(*d x*) ⊓ *x* = *bot*
proof −
have ∀ *x*. !(*d x*) ⊓ *d x* = *bot*
by (*simp add*: *d-closed neg-char*)
thus *?thesis*

by (*metis d1-eq il-inf-associative il-sub-inf-left-zero*)
qed

lemma *d-isotone*:
assumes $x \leq y$
shows $d\ x \leq d\ y$
proof −
obtain *ad* **where** *1*: $\forall x.$ *are-complementary* $(d\ x)$ $(ad\ x)$
using *d-closed* **by** *moura*
hence $ad\ y \sqcap x \leq bot$
by (*metis assms d1-eq il-inf-associative il-sub-inf-left-zero il-sub-inf-right-isotone*)
thus *?thesis*
using *1* **by** (*metis d2 il-bot-unit il-inf-left-unit il-inf-right-dist-sup il-less-eq*)
qed

lemma *d-isotone-var*:
$d\ x \leq d\ (x \sqcup y)$
using *d-isotone il-associative il-less-eq* **by** *auto*

lemma *d3-conv*:
$d\ (x \sqcap y) \leq d\ (x \sqcap d\ y)$
by (*metis* (*mono-tags*, *hide-lams*) *d1-eq d2 d-closed il-inf-associative*)

lemma *d-test-inf-idempotent*:
$d\ x \sqcap d\ x = d\ x$
by (*metis d-idempotent d1-eq*)

lemma *d-test-inf-closed*:
assumes *test x*
and *test y*
shows $d\ (x \sqcap y) = x \sqcap y$
proof (*rule antisym*)
have $d\ (x \sqcap y) = d\ (x \sqcap y) \sqcap d\ (x \sqcap y)$
by (*simp add*: *d-test-inf-idempotent*)
also have ... $\leq x \sqcap d\ (x \sqcap y)$
by (*simp add*: *assms*(*1*) *d2 il-sub-inf-left-isotone*)
also have ... $\leq x \sqcap y$
by (*metis assms d-isotone il-sub-inf-right-isotone test-inf-left-below test-d-fixpoint*)
finally show $d\ (x \sqcap y) \leq x \sqcap y$
.
show $x \sqcap y \leq d\ (x \sqcap y)$
using *assms d-increasing-sub-identity dual-order.trans test-inf-left-below test-sub-identity* **by** *blast*
qed

lemma *test-inf-closed*:
test x $\Longrightarrow$ *test y* $\Longrightarrow$ *test* $(x \sqcap y)$

using *d-test-inf-closed test-d-fixpoint-iff* **by** *simp*

lemma *test-sup-closed*:
test $x \Longrightarrow$ *test* $y \Longrightarrow$ *test* $(x \sqcup y)$
using *test-inf-closed test-inf-closed-sup-closed* **by** *simp*

lemma *d-export*:
assumes *test* x
shows $d\ (x \sqcap y) = x \sqcap d\ y$
proof (*rule antisym*)
have *1*: $d\ (x \sqcap y) \leq x$
by (*simp add*: *assms d2*)
have $d\ (x \sqcap y) \leq d\ y$
by (*metis assms d-isotone-var il-inf-left-unit il-inf-right-dist-sup*)
thus $d\ (x \sqcap y) \leq x \sqcap d\ y$
using *1* **by** (*metis assms d-idempotent llp dual-order.trans*
il-sub-inf-right-isotone)
have $y = (!x \sqcap y) \sqcup (x \sqcap y)$
by (*metis assms il-commutative il-inf-left-unit il-inf-right-dist-sup neg-char*)
also have ... $= (!x \sqcap y) \sqcup (d\ (x \sqcap y) \sqcap x \sqcap y)$
by (*metis d1-eq il-inf-associative*)
also have ... $= (!x \sqcap y) \sqcup (d\ (x \sqcap y) \sqcap y)$
using *1* **by** (*smt calculation d1-eq il-associative il-commutative*
il-inf-associative il-inf-right-dist-sup il-less-eq il-sub-inf-right-isotone-var)
also have ... $= (!x \sqcup d\ (x \sqcap y)) \sqcap y$
by (*simp add*: *il-inf-right-dist-sup*)
finally have $y \leq (!x \sqcup d\ (x \sqcap y)) \sqcap y$
by *simp*
hence $d\ y \leq\ !x \sqcup d\ (x \sqcap y)$
using *assms llp test-sup-closed neg-test d-closed* **by** *simp*
hence $d\ y \sqcap x \leq d\ (x \sqcap y)$
by (*simp add*: *assms d-closed test-shunting*)
thus $x \sqcap d\ y \leq d\ (x \sqcap y)$
by (*metis assms d-closed test-inf-commutative*)
qed

lemma *test-inf-left-dist-sup*:
assumes *test* x
and *test* y
and *test* z
shows $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$
proof −
have $x \sqcap (y \sqcup z) = (y \sqcup z) \sqcap x$
using *assms test-sup-closed test-inf-commutative* **by** *smt*
also have ... $= (y \sqcap x) \sqcup (z \sqcap x)$
using *il-inf-right-dist-sup* **by** *simp*
also have ... $= (x \sqcap y) \sqcup (x \sqcap z)$
using *assms test-sup-closed test-inf-commutative* **by** *smt*
finally show *?thesis*

```
  .
qed

lemma !x ⊔ !y = !(!(!x ⊔ !y)) nitpick [expect=genuine] oops
lemma d x = !(!x) nitpick [expect=genuine] oops

sublocale subset-boolean-algebra where uminus = λ x . !(d x)
proof
  show ⋀x y z. !(d x) ⊔ (!(d y) ⊔ !(d z)) = !(d x) ⊔ !(d y) ⊔ !(d z)
    using il-associative by blast
  show ⋀x y. !(d x) ⊔ !(d y) = !(d y) ⊔ !(d x)
    by (simp add: il-commutative)
  show ⋀x y. !(d x) ⊔ !(d y) = !(d (!(d (!(d x) ⊔ !(d y)))))
  proof −
    fix x y
    have test (!(d x)) ∧ test (!(d y))
      by (simp add: d-closed neg-test)
    hence test (!(d x) ⊔ !(d y))
      by (simp add: test-sup-closed)
    thus !(d x) ⊔ !(d y) = !(d (!(d (!(d x) ⊔ !(d y)))))
      by (simp add: d-neg neg-involutive test-d-fixpoint)
  qed
  show ⋀x y. !(d x) = !(d (!(d (!(d x))) ⊔ !(d y))) ⊔ !(d (!(d (!(d x))) ⊔ !(d (!(d
y)))))
  proof −
    fix x y
    have !(d (!(d (!(d x))) ⊔ !(d y))) ⊔ !(d (!(d (!(d x))) ⊔ !(d (!(d y))))) = !(d x
⊔ !(d y)) ⊔ !(d x ⊔ d y)
      using d-closed neg-test test-sup-closed neg-involutive test-d-fixpoint by auto
    also have ... = (!(d x) ⊓ d y) ⊔ (!(d x) ⊓ !(d y))
      using d-closed neg-test test-sup-closed neg-involutive de-morgan-2 by auto
    also have ... = !(d x) ⊓ (d y ⊔ !(d y))
      using d-closed neg-test test-inf-left-dist-sup by auto
    also have ... = !(d x) ⊓ top
      by (simp add: neg-char d-closed)
    finally show !(d x) = !(d (!(d (!(d x))) ⊔ !(d y))) ⊔ !(d (!(d (!(d x))) ⊔ !(d
(!(d y)))))
      by simp
  qed
qed

lemma d-dist-sup:
  d (x ⊔ y) = d x ⊔ d y
proof (rule antisym)
  have x ≤ d x ⊓ x
    by (simp add: d1)
  also have ... ≤ (d x ⊔ d y) ⊓ (x ⊔ y)
    using il-associative il-inf-right-dist-sup il-less-eq il-sub-inf-right-isotone by
auto
```

finally have *1*: $x \le (d\ x \sqcup d\ y) \sqcap (x \sqcup y)$
  .
have $y \le d\ y \sqcap y$
  by (*simp add*: *d1*)
also have ... $\le (d\ y \sqcup d\ x) \sqcap (y \sqcup x)$
  using *il-associative il-idempotent il-inf-right-dist-sup il-less-eq il-sub-inf-right-isotone* **by** *simp*
finally have $y \le (d\ x \sqcup d\ y) \sqcap (x \sqcup y)$
  using *il-commutative* **by** *auto*
hence $x \sqcup y \le (d\ x \sqcup d\ y) \sqcap (x \sqcup y)$
  using *1* **by** (*metis il-associative il-less-eq*)
thus $d\ (x \sqcup y) \le d\ x \sqcup d\ y$
  using *llp test-sup-closed neg-test d-closed* **by** *simp*
show $d\ x \sqcup d\ y \le d\ (x \sqcup y)$
  using *d-isotone-var il-associative il-commutative il-less-eq* **by** *fastforce*
qed

end

class *pd-semiring-extended* = *pd-semiring* + *uminus* +
  assumes *uminus-def*: $-x = !(d\ x)$
begin

subclass *subset-boolean-algebra*
  by (*metis subset-boolean-algebra-axioms uminus-def ext*)

end

7.4 Domain Semirings

class *d-semiring* = *pd-semiring* +
  assumes *d3*: $d\ (x \sqcap d\ y) \le d\ (x \sqcap y)$
begin

lemma *d3-eq*: $d\ (x \sqcap d\ y) = d\ (x \sqcap y)$
  by (*simp add*: *antisym d3 d3-conv*)

end

Axioms (d1), (d2) and (d3) are independent in IL-semirings.

context *il-semiring*
begin

context
  fixes $d :: 'a \Rightarrow 'a$
  assumes *d-closed*: *test* $(d\ x)$
begin

context
  assumes *d1*: $x \le d\ x \sqcap x$

assumes *d2*: *test p* $\Longrightarrow$ *d* $(p \sqcap x) \leq p$
begin

lemma *d3*: *d* $(x \sqcap d\ y) \leq d\ (x \sqcap y)$ **nitpick** [*expect=genuine*] **oops**

end

context
assumes *d1*: $x \leq d\ x \sqcap x$
assumes *d3*: *d* $(x \sqcap d\ y) \leq d\ (x \sqcap y)$
begin

lemma *d2*: *test p* $\Longrightarrow$ *d* $(p \sqcap x) \leq p$ **nitpick** [*expect=genuine*] **oops**

end

context
assumes *d2*: *test p* $\Longrightarrow$ *d* $(p \sqcap x) \leq p$
assumes *d3*: *d* $(x \sqcap d\ y) \leq d\ (x \sqcap y)$
begin

lemma *d1*: $x \leq d\ x \sqcap x$ **nitpick** [*expect=genuine*] **oops**

end

end

end

class *d-semiring-var* = *ppd-semiring* +
assumes *d3-var*: *d* $(x \sqcap d\ y) \leq d\ (x \sqcap y)$
assumes *d-strict-eq-var*: *d bot* = *bot*
begin

lemma *d2-var*:
assumes *test p*
shows *d* $(p \sqcap x) \leq p$
proof −
have !$p \sqcap p \sqcap x$ = *bot*
by (*simp add*: *assms neg-char*)
hence *d* (!$p \sqcap p \sqcap x$) = *bot*
by (*simp add*: *d-strict-eq-var*)
hence *d* (!$p \sqcap d\ (p \sqcap x)$) = *bot*
by (*metis d3-var il-inf-associative less-eq-bot*)
hence !$p \sqcap d\ (p \sqcap x)$ = *bot*
using *d-bot-only* **by** *blast*
thus *?thesis*
by (*metis* (*no-types*, *hide-lams*) *assms d-sub-identity il-bot-unit il-inf-left-unit il-inf-right-dist-sup il-inf-right-unit il-sub-inf-right-isotone neg-char*)

qed

subclass *d-semiring*
proof
  show $\bigwedge p\ x.\ test\ p \Longrightarrow d\ (p \sqcap x) \leq p$
    by (*simp add*: *d2-var*)
  show $\bigwedge x\ y.\ d\ (x \sqcap d\ y) \leq d\ (x \sqcap y)$
    by (*simp add*: *d3-var*)
qed

end

8 Antidomain Semirings

We now develop prepreantidomain semirings, preantidomain semirings and antidomain semirings. See [6, 7, 8] for related work on internal axioms for antidomain.

8.1 Prepreantidomain Semirings

Definition 20

class *ppa-semiring* = *il-semiring* + *uminus* +
  assumes *a-inf-complement-bot*: $-x \sqcap x = bot$
  assumes *a-stone*[*simp*]: $-x \sqcup --x = top$
begin

Theorem 21

lemma *l1*:
  $-top = bot$
  by (*metis a-inf-complement-bot il-inf-right-unit*)

lemma *l2*:
  $-bot = top$
  by (*metis l1 a-stone il-unit-bot*)

lemma *l3*:
  $-x \leq -y \Longrightarrow -x \sqcap y = bot$
  by (*metis a-inf-complement-bot il-bot-unit il-inf-right-dist-sup il-less-eq*)

lemma *l5*:
  $--x \leq --y \Longrightarrow -y \leq -x$
  by (*metis* (*mono-tags*, *hide-lams*) *l3 a-stone bot-least il-bot-unit il-inf-left-unit il-inf-right-dist-sup il-inf-right-unit il-sub-inf-right-isotone sup-right-isotone*)

lemma *l4*:
  $---x = -x$
  by (*metis l5 a-inf-complement-bot a-stone antisym bot-least il-inf-left-unit il-inf-right-dist-sup il-inf-right-unit il-sub-inf-right-isotone il-unit-bot*)

lemma *l6*:
$-x \sqcap --x = bot$
by (*metis l3 l5 a-inf-complement-bot a-stone il-inf-left-unit il-inf-right-dist-sup il-inf-right-unit il-less-eq il-sub-inf-right-isotone il-unit-bot*)

lemma *l7*:
$-x \sqcap -y = -y \sqcap -x$
using *l6 a-inf-complement-bot a-stone test-inf-commutative* **by** *blast*

lemma *l8*:
$x \leq --x \sqcap x$
by (*metis a-inf-complement-bot a-stone il-idempotent il-inf-left-unit il-inf-right-dist-sup il-less-eq il-unit-bot*)

sublocale *ppa-ppd*: *ppd-semiring* **where** $d = \lambda x\ .\ --x$
proof
show $\bigwedge x.$ *test* $(- - x)$
using *l4 l6* **by** *force*
show $\bigwedge x.\ x \leq - - x \sqcap x$
by (*simp add*: *l8*)
qed

end

8.2 Preantidomain Semirings

Definition 22

class *pa-semiring* = *ppa-semiring* +
assumes *pad2*: $--x \leq -(-x \sqcap y)$
begin

Theorem 23

lemma *l10*:
$-x \sqcap y = bot \Longrightarrow -x \leq -y$
by (*metis a-stone il-inf-left-unit il-inf-right-dist-sup il-unit-bot l4 pad2*)

lemma *l10-iff*:
$-x \sqcap y = bot \longleftrightarrow -x \leq -y$
using *l10 l3* **by** *blast*

lemma *l13*:
$--(--x \sqcap y) \leq --x$
by (*metis l4 l5 pad2*)

lemma *l14*:
$-(x \sqcap --y) \leq -(x \sqcap y)$

by (*metis il-inf-associative l4 pad2 ppa-ppd.d1-eq*)

lemma *l9*:
$x \leq y \Longrightarrow -y \leq -x$
by (*metis l10 a-inf-complement-bot il-commutative il-less-eq il-sub-inf-right-isotone il-unit-bot*)

lemma *l11*:
$- x \sqcup - y = - (- - x \sqcap - - y)$
proof −
have *1*: $\bigwedge x\ y\ .\ x \leq y \longleftrightarrow x \sqcup y = y$
by (*simp add*: *il-less-eq*)
have *4*: $\bigwedge x\ y\ .\ \neg(x \leq y) \vee x \sqcup y = y$
using *1* **by** *metis*
have *5*: $\bigwedge x\ y\ z\ .\ (x \sqcap y) \sqcup (x \sqcap z) \leq x \sqcap (y \sqcup z)$
by (*simp add*: *il-sub-inf-right-isotone-var*)
have *6*: $\bigwedge x\ y\ .\ - - x \leq - (- x \sqcap y)$
by (*simp add*: *pad2*)
have *7*: $\bigwedge x\ y\ z\ .\ x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$
by (*simp add*: *il-associative*)
have *8*: $\bigwedge x\ y\ z\ .\ (x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$
using *7* **by** *metis*
have *9*: $\bigwedge x\ y\ .\ x \sqcup y = y \sqcup x$
by (*simp add*: *il-commutative*)
have *10*: $\bigwedge x\ .\ x \sqcup bot = x$
by (*simp add*: *il-bot-unit*)
have *11*: $\bigwedge x\ .\ x \sqcup x = x$
by *simp*
have *12*: $\bigwedge x\ y\ z\ .\ x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$
by (*simp add*: *il-inf-associative*)
have *13*: $\bigwedge x\ y\ z\ .\ (x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$
using *12* **by** *metis*
have *14*: $\bigwedge x\ .\ top \sqcap x = x$
by *simp*
have *15*: $\bigwedge x\ .\ x \sqcap top = x$
by *simp*
have *16*: $\bigwedge x\ y\ z\ .\ (x \sqcup y) \sqcap z = (x \sqcap z) \sqcup (y \sqcap z)$
by (*simp add*: *il-inf-right-dist-sup*)
have *17*: $\bigwedge x\ y\ z\ .\ (x \sqcap y) \sqcup (z \sqcap y) = (x \sqcup z) \sqcap y$
using *16* **by** *metis*
have *18*: $\bigwedge x\ .\ bot \sqcap x = bot$
by *simp*
have *19*: $\bigwedge x\ .\ - x \sqcup - - x = top$
by *simp*
have *20*: $\bigwedge x\ .\ - x \sqcap x = bot$
by (*simp add*: *a-inf-complement-bot*)
have *23*: $\bigwedge x\ y\ z\ .\ ((x \sqcap y) \sqcup (x \sqcap z)) \sqcup (x \sqcap (y \sqcup z)) = x \sqcap (y \sqcup z)$
using *4 5* **by** *metis*
have *24*: $\bigwedge x\ y\ z\ .\ (x \sqcap (y \sqcup z)) \sqcup ((x \sqcap y) \sqcup (x \sqcap z)) = x \sqcap (y \sqcup z)$

using *9 23* **by** *metis*
have *25*: $\bigwedge x\ y\ .\ -\ -\ x \sqcup -\ (-\ x \sqcap y) = -\ (-\ x \sqcap y)$
using *4 6* **by** *metis*
have *26*: $\bigwedge x\ y\ z\ .\ x \sqcup (y \sqcup z) = y \sqcup (x \sqcup z)$
using *8 9* **by** *metis*
have *27*: $\bigwedge x\ y\ z\ .\ (x \sqcap y) \sqcup ((x \sqcap z) \sqcup (x \sqcap (y \sqcup z))) = x \sqcap (y \sqcup z)$
using *9 24 26* **by** *metis*
have *30*: $\bigwedge x\ .\ bot \sqcup x = x$
using *9 10* **by** *metis*
have *31*: $\bigwedge x\ y\ .\ x \sqcup (x \sqcup y) = x \sqcup y$
using *8 11* **by** *metis*
have *34*: $\bigwedge u\ x\ y\ z\ .\ ((x \sqcup y) \sqcap z) \sqcup u = (x \sqcap z) \sqcup ((y \sqcap z) \sqcup u)$
using *8 17* **by** *metis*
have *35*: $\bigwedge u\ x\ y\ z\ .\ (x \sqcap (y \sqcap z)) \sqcup (u \sqcap z) = ((x \sqcap y) \sqcup u) \sqcap z$
using *13 17* **by** *metis*
have *36*: $\bigwedge u\ x\ y\ z\ .\ (x \sqcap y) \sqcup (z \sqcap (u \sqcap y)) = (x \sqcup (z \sqcap u)) \sqcap y$
using *13 17* **by** *metis*
have *39*: $\bigwedge x\ y\ .\ -\ x \sqcup (-\ -\ x \sqcup y) = top \sqcup y$
using *8 19* **by** *metis*
have *41*: $\bigwedge x\ y\ .\ -\ x \sqcap (x \sqcap y) = bot$
using *13 18 20* **by** *metis*
have *42*: $-\ top = bot$
using *15 20* **by** *metis*
have *43*: $\bigwedge x\ y\ .\ (-\ x \sqcup y) \sqcap x = y \sqcap x$
using *17 20 30* **by** *metis*
have *44*: $\bigwedge x\ y\ .\ (x \sqcup -\ y) \sqcap y = x \sqcap y$
using *9 17 20 30* **by** *metis*
have *46*: $\bigwedge x\ .\ -\ bot \sqcup -\ -\ x = -\ bot$
using *9 20 25* **by** *metis*
have *50*: $-\ bot = top$
using *19 30 42* **by** *metis*
have *51*: $\bigwedge x\ .\ top \sqcup -\ -\ x = top$
using *46 50* **by** *metis*
have *63*: $\bigwedge x\ y\ .\ x \sqcup ((x \sqcap -\ y) \sqcup (x \sqcap -\ -\ y)) = x$
using *9 15 19 26 27* **by** *metis*
have *66*: $\bigwedge x\ y\ .\ (-\ (x \sqcup y) \sqcap x) \sqcup (-\ (x \sqcup y) \sqcap y) = bot$
using *9 20 27 30* **by** *metis*
have *67*: $\bigwedge x\ y\ z\ .\ (x \sqcap -\ -\ y) \sqcup (x \sqcap -\ (-\ y \sqcap z)) = x \sqcap -\ (-\ y \sqcap z)$
using *11 25 27* **by** *metis*
have *70*: $\bigwedge x\ y\ .\ x \sqcup (x \sqcap -\ -\ y) = x$
using *9 15 27 31 51* **by** *metis*
have *82*: $\bigwedge x\ .\ top \sqcup -\ x = top$
using *9 19 31* **by** *metis*
have *89*: $\bigwedge x\ y\ .\ x \sqcup (-\ y \sqcap x) = x$
using *14 17 82* **by** *metis*
have *102*: $\bigwedge x\ y\ z\ .\ x \sqcup (y \sqcup (x \sqcap -\ -\ z)) = y \sqcup x$
using *26 70* **by** *metis*
have *104*: $\bigwedge x\ y\ .\ x \sqcup (x \sqcap -\ y) = x$
using *9 63 102* **by** *metis*

have *112*: $\bigwedge x\ y\ z\ .\ (-\ x \sqcap y) \sqcup ((-\ -\ x \sqcap y) \sqcup z) = y \sqcup z$
using *14 19 34* **by** *metis*
have *117*: $\bigwedge x\ y\ z\ .\ x \sqcup ((x \sqcap -\ y) \sqcup z) = x \sqcup z$
using *8 104* **by** *metis*
have *120*: $\bigwedge x\ y\ z\ .\ x \sqcup (y \sqcup (x \sqcap -\ z)) = y \sqcup x$
using *26 104* **by** *metis*
have *124*: $\bigwedge x\ .\ -\ -\ x \sqcap x = x$
using *14 19 43* **by** *metis*
have *128*: $\bigwedge x\ y\ .\ -\ -\ x \sqcap (x \sqcap y) = x \sqcap y$
using *13 124* **by** *metis*
have *131*: $\bigwedge x\ .\ -\ x \sqcup -\ -\ -\ x = -\ x$
using *9 25 124* **by** *metis*
have *133*: $\bigwedge x\ .\ -\ -\ -\ x = -\ x$
using *9 104 124 131* **by** *metis*
have *135*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ (-\ -\ x \sqcap y) = -\ (-\ -\ x \sqcap y)$
using *25 133* **by** *metis*
have *137*: $\bigwedge x\ y\ .\ (-\ x \sqcup y) \sqcap -\ -\ x = y \sqcap -\ -\ x$
using *43 133* **by** *metis*
have *145*: $\bigwedge x\ y\ z\ .\ ((-\ (x \sqcap y) \sqcap x) \sqcup z) \sqcap y = z \sqcap y$
using *20 30 35* **by** *metis*
have *183*: $\bigwedge x\ y\ z\ .\ (x \sqcup (-\ -\ (y \sqcap z) \sqcap y)) \sqcap z = (x \sqcup y) \sqcap z$
using *17 36 124* **by** *metis*
have *289*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ (-\ x \sqcap y) = \mathit{top}$
using *25 39 82* **by** *metis*
have *316*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcap y) \sqcap x = x$
using *14 43 289* **by** *metis*
have *317*: $\bigwedge x\ y\ z\ .\ -\ (-\ x \sqcap y) \sqcap (x \sqcap z) = x \sqcap z$
using *13 316* **by** *metis*
have *320*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ -\ (-\ x \sqcap y) = -\ x$
using *9 25 316* **by** *metis*
have *321*: $\bigwedge x\ y\ .\ -\ -\ (-\ x \sqcap y) \sqcap x = \mathit{bot}$
using *41 316* **by** *metis*
have *374*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ (x \sqcap y) = -\ (x \sqcap y)$
using *25 128 133* **by** *metis*
have *388*: $\bigwedge x\ y\ .\ -\ (x \sqcap y) \sqcap -\ x = -\ x$
using *128 316* **by** *metis*
have *389*: $\bigwedge x\ y\ .\ -\ -\ (x \sqcap y) \sqcap -\ x = \mathit{bot}$
using *128 321* **by** *metis*
have *405*: $\bigwedge x\ y\ z\ .\ -\ (x \sqcap y) \sqcap (-\ x \sqcap z) = -\ x \sqcap z$
using *13 388* **by** *metis*
have *406*: $\bigwedge x\ y\ z\ .\ -\ (x \sqcap (y \sqcap z)) \sqcap -\ (x \sqcap y) = -\ (x \sqcap y)$
using *13 388* **by** *metis*
have *420*: $\bigwedge x\ y\ .\ -\ x \sqcap -\ -\ (-\ x \sqcap y) = -\ -\ (-\ x \sqcap y)$
using *316 388* **by** *metis*
have *422*: $\bigwedge x\ y\ z\ .\ -\ -\ (x \sqcap y) \sqcap (-\ x \sqcap z) = \mathit{bot}$
using *13 18 389* **by** *metis*
have *758*: $\bigwedge x\ y\ z\ .\ x \sqcup (x \sqcap (-\ y \sqcap -\ z)) = x$
using *13 104 117* **by** *metis*
have *1092*: $\bigwedge x\ y\ .\ -\ (x \sqcup y) \sqcap x = \mathit{bot}$

using *9 30 31 66* **by** *metis*
have *1130*: $\bigwedge x\ y\ z\ .\ (-\ (x \sqcup y) \sqcup z) \sqcap x = z \sqcap x$
using *17 30 1092* **by** *metis*
have *1156*: $\bigwedge x\ y\ .\ -\ -\ x \sqcap -\ (-\ x \sqcap y) = -\ -\ x$
using *67 104 124 133* **by** *metis*
have *2098*: $\bigwedge x\ y\ .\ -\ -\ (x \sqcup y) \sqcap x = x$
using *14 19 1130* **by** *metis*
have *2125*: $\bigwedge x\ y\ .\ -\ -\ (x \sqcup y) \sqcap y = y$
using *9 2098* **by** *metis*
have *2138*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ -\ (x \sqcup y) = top$
using *9 289 2098* **by** *metis*
have *2139*: $\bigwedge x\ y\ .\ -\ x \sqcap -\ (x \sqcup y) = -\ (x \sqcup y)$
using *316 2098* **by** *metis*
have *2192*: $\bigwedge x\ y\ .\ -\ -\ x \sqcap (-\ y \sqcap x) = -\ y \sqcap x$
using *89 2125* **by** *metis*
have *2202*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ -\ (y \sqcup x) = top$
using *9 289 2125* **by** *metis*
have *2344*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcap y) \sqcup -\ -\ y = top$
using *89 2202* **by** *metis*
have *2547*: $\bigwedge x\ y\ z\ .\ -\ x \sqcup ((-\ -\ x \sqcap -\ y) \sqcup z) = -\ x \sqcup (-\ y \sqcup z)$
using *112 117* **by** *metis*
have *3023*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ (-\ y \sqcap -\ x) = top$
using *9 133 2344* **by** *metis*
have *3134*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcap -\ y) \sqcap y = y$
using *14 43 3023* **by** *metis*
have *3135*: $\bigwedge x\ y\ .\ -\ x \sqcap (-\ y \sqcap -\ x) = -\ y \sqcap -\ x$
using *14 44 3023* **by** *metis*
have *3962*: $\bigwedge x\ y\ .\ -\ -\ (x \sqcup y) \sqcap -\ -\ x = -\ -\ x$
using *14 137 2138* **by** *metis*
have *5496*: $\bigwedge x\ y\ z\ .\ -\ -\ (x \sqcap y) \sqcap -\ (x \sqcup z) = bot$
using *422 2139* **by** *metis*
have *9414*: $\bigwedge x\ y\ .\ -\ -\ (-\ x \sqcap y) \sqcap y = -\ x \sqcap y$
using *9 104 183 320* **by** *metis*
have *9520*: $\bigwedge x\ y\ z\ .\ -\ -\ (-\ x \sqcap y) \sqcap -\ -\ (x \sqcap z) = bot$
using *374 5496* **by** *metis*
have *11070*: $\bigwedge x\ y\ z\ .\ -\ (-\ -\ x \sqcap y) \sqcup (-\ x \sqcap -\ z) = -\ (-\ -\ x \sqcap y)$
using *317 758* **by** *metis*
have *12371*: $\bigwedge x\ y\ .\ -\ x \sqcap -\ (-\ -\ x \sqcap y) = -\ x$
using *133 1156* **by** *metis*
have *12377*: $\bigwedge x\ y\ .\ -\ x \sqcap -\ (x \sqcap y) = -\ x$
using *128 133 1156* **by** *metis*
have *12384*: $\bigwedge x\ y\ .\ -\ (x \sqcup y) \sqcap -\ y = -\ (x \sqcup y)$
using *133 1156 2125* **by** *metis*
have *12394*: $\bigwedge x\ y\ .\ -\ -\ (-\ x \sqcap -\ y) = -\ x \sqcap -\ y$
using *1156 3134 9414* **by** *metis*
have *12640*: $\bigwedge x\ y\ .\ -\ x \sqcap -\ (-\ y \sqcap x) = -\ x$
using *89 12384* **by** *metis*
have *24648*: $\bigwedge x\ y\ .\ (-\ x \sqcap -\ y) \sqcup -\ (-\ x \sqcap -\ y) = top$
using *19 12394* **by** *metis*

have *28270*: $\bigwedge x\ y\ z\ .\ -\ -\ (x \sqcap y) \sqcup -\ (-\ x \sqcap z) = -\ (-\ x \sqcap z)$
  using *374 405* **by** *metis*
have *28339*: $\bigwedge x\ y\ .\ -\ (-\ -\ (x \sqcap y) \sqcap x) = -\ (x \sqcap y)$
  using *124 406 12371* **by** *metis*
have *28423*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcap -\ y) = -\ (-\ y \sqcap -\ x)$
  using *13 3135 12394 28339* **by** *metis*
have *28487*: $\bigwedge x\ y\ .\ -\ x \sqcap -\ y = -\ y \sqcap -\ x$
  using *2098 3962 12394 28423* **by** *metis*
have *52423*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcap -\ (-\ x \sqcap y)) \sqcap y = y$
  using *14 145 24648 28487* **by** *metis*
have *52522*: $\bigwedge x\ y\ .\ -\ x \sqcap -\ (-\ x \sqcap y) = -\ x \sqcap -\ y$
  using *13 12377 12394 12640 28487 52423* **by** *metis*
have *61103*: $\bigwedge x\ y\ z\ .\ -\ (-\ -\ x \sqcap y) \sqcup z = -\ x \sqcup (-\ y \sqcup z)$
  using *112 2547 12371 52522* **by** *metis*
have *61158*: $\bigwedge x\ y\ .\ -\ -\ (-\ x \sqcap y) = -\ x \sqcap -\ -\ y$
  using *420 52522* **by** *metis*
have *61231*: $\bigwedge x\ y\ z\ .\ -\ x \sqcap (-\ -\ y \sqcap -\ (x \sqcap z)) = -\ x \sqcap -\ -\ y$
  using *13 15 50 133 9520 52522 61158* **by** *metis*
have *61313*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ y = -\ (-\ -\ y \sqcap x)$
  using *120 11070 61103* **by** *metis*
have *61393*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcap -\ -\ y) = -\ (-\ x \sqcap y)$
  using *13 28270 61158 61231 61313* **by** *metis*
have *61422*: $\bigwedge x\ y\ .\ -\ (-\ -\ x \sqcap y) = -\ (-\ -\ y \sqcap x)$
  using *13 135 2192 61158 61313* **by** *metis*
show *?thesis*
  using *61313 61393 61422* **by** *metis*
qed

lemma *l12*:
$-\ x \sqcap -\ y = -\ (x \sqcup y)$
proof –
have *1*: $\bigwedge x\ y\ .\ x \leq y \longleftrightarrow x \sqcup y = y$
  by (*simp add*: *il-less-eq*)
have *4*: $\bigwedge x\ y\ .\ \neg(x \leq y) \vee x \sqcup y = y$
  using *1* **by** *metis*
have *5*: $\bigwedge x\ y\ z\ .\ (x \sqcap y) \sqcup (x \sqcap z) \leq x \sqcap (y \sqcup z)$
  by (*simp add*: *il-sub-inf-right-isotone-var*)
have *6*: $\bigwedge x\ y\ .\ -\ -\ x \leq -\ (-\ x \sqcap y)$
  by (*simp add*: *pad2*)
have *7*: $\bigwedge x\ y\ z\ .\ x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$
  by (*simp add*: *il-associative*)
have *8*: $\bigwedge x\ y\ z\ .\ (x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$
  using *7* **by** *metis*
have *9*: $\bigwedge x\ y\ .\ x \sqcup y = y \sqcup x$
  by (*simp add*: *il-commutative*)
have *10*: $\bigwedge x\ .\ x \sqcup bot = x$
  by (*simp add*: *il-bot-unit*)
have *11*: $\bigwedge x\ .\ x \sqcup x = x$
  by *simp*

have *12*: $\bigwedge x\ y\ z\ .\ x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$
by (*simp add*: *il-inf-associative*)
have *13*: $\bigwedge x\ y\ z\ .\ (x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$
using *12* **by** *metis*
have *14*: $\bigwedge x\ .\ top \sqcap x = x$
by *simp*
have *15*: $\bigwedge x\ .\ x \sqcap top = x$
by *simp*
have *16*: $\bigwedge x\ y\ z\ .\ (x \sqcup y) \sqcap z = (x \sqcap z) \sqcup (y \sqcap z)$
by (*simp add*: *il-inf-right-dist-sup*)
have *17*: $\bigwedge x\ y\ z\ .\ (x \sqcap y) \sqcup (z \sqcap y) = (x \sqcup z) \sqcap y$
using *16* **by** *metis*
have *18*: $\bigwedge x\ .\ bot \sqcap x = bot$
by *simp*
have *19*: $\bigwedge x\ .\ -\ x \sqcup -\ -\ x = top$
by *simp*
have *20*: $\bigwedge x\ .\ -\ x \sqcap x = bot$
by (*simp add*: *a-inf-complement-bot*)
have *22*: $\bigwedge x\ y\ z\ .\ ((x \sqcap y) \sqcup (x \sqcap z)) \sqcup (x \sqcap (y \sqcup z)) = x \sqcap (y \sqcup z)$
using *4 5* **by** *metis*
have *23*: $\bigwedge x\ y\ z\ .\ (x \sqcap (y \sqcup z)) \sqcup ((x \sqcap y) \sqcup (x \sqcap z)) = x \sqcap (y \sqcup z)$
using *9 22* **by** *metis*
have *24*: $\bigwedge x\ y\ .\ -\ -\ x \sqcup -\ (-\ x \sqcap y) = -\ (-\ x \sqcap y)$
using *4 6* **by** *metis*
have *25*: $\bigwedge x\ y\ z\ .\ x \sqcup (y \sqcup z) = y \sqcup (x \sqcup z)$
using *8 9* **by** *metis*
have *26*: $\bigwedge x\ y\ z\ .\ (x \sqcap y) \sqcup ((x \sqcap z) \sqcup (x \sqcap (y \sqcup z))) = x \sqcap (y \sqcup z)$
using *9 23 25* **by** *metis*
have *29*: $\bigwedge x\ .\ bot \sqcup x = x$
using *9 10* **by** *metis*
have *30*: $\bigwedge x\ y\ .\ x \sqcup (x \sqcup y) = x \sqcup y$
using *8 11* **by** *metis*
have *32*: $\bigwedge x\ y\ .\ x \sqcup (y \sqcup x) = y \sqcup x$
using *8 9 11* **by** *metis*
have *33*: $\bigwedge u\ x\ y\ z\ .\ ((x \sqcup y) \sqcap z) \sqcup u = (x \sqcap z) \sqcup ((y \sqcap z) \sqcup u)$
using *8 17* **by** *metis*
have *34*: $\bigwedge u\ x\ y\ z\ .\ (x \sqcap (y \sqcap z)) \sqcup (u \sqcap z) = ((x \sqcap y) \sqcup u) \sqcap z$
using *13 17* **by** *metis*
have *35*: $\bigwedge u\ x\ y\ z\ .\ (x \sqcap y) \sqcup (z \sqcap (u \sqcap y)) = (x \sqcup (z \sqcap u)) \sqcap y$
using *13 17* **by** *metis*
have *36*: $\bigwedge x\ y\ .\ (top \sqcup x) \sqcap y = y \sqcup (x \sqcap y)$
using *14 17* **by** *metis*
have *37*: $\bigwedge x\ y\ .\ (x \sqcup top) \sqcap y = y \sqcup (x \sqcap y)$
using *9 14 17* **by** *metis*
have *38*: $\bigwedge x\ y\ .\ -\ x \sqcup (-\ -\ x \sqcup y) = top \sqcup y$
using *8 19* **by** *metis*
have *40*: $\bigwedge x\ y\ .\ -\ x \sqcap (x \sqcap y) = bot$
using *13 18 20* **by** *metis*
have *41*: $-\ top = bot$

using *15 20* **by** *metis*
have *42*: $\bigwedge x\ y\ .\ (-\ x \sqcup y) \sqcap x = y \sqcap x$
using *17 20 29* **by** *metis*
have *43*: $\bigwedge x\ y\ .\ (x \sqcup -\ y) \sqcap y = x \sqcap y$
using *9 17 20 29* **by** *metis*
have *45*: $\bigwedge x\ .\ -\ bot \sqcup -\ -\ x = -\ bot$
using *9 20 24* **by** *metis*
have *46*: $\bigwedge u\ x\ y\ z\ .\ (x \sqcap y) \sqcup (z \sqcup (u \sqcap y)) = z \sqcup ((x \sqcup u) \sqcap y)$
using *17 25* **by** *metis*
have *47*: $\bigwedge x\ y\ .\ -\ x \sqcup (y \sqcup -\ -\ x) = y \sqcup top$
using *19 25* **by** *metis*
have *49*: $-\ bot = top$
using *19 29 41* **by** *metis*
have *50*: $\bigwedge x\ .\ top \sqcup -\ -\ x = top$
using *45 49* **by** *metis*
have *54*: $\bigwedge u\ x\ y\ z\ .\ (x \sqcap y) \sqcup ((x \sqcap z) \sqcup ((x \sqcap (y \sqcup z)) \sqcup u)) = (x \sqcap (y \sqcup z)) \sqcup u$
using *8 26* **by** *metis*
have *58*: $\bigwedge u\ x\ y\ z\ .\ (x \sqcap (y \sqcap z)) \sqcup ((x \sqcap (y \sqcap u)) \sqcup (x \sqcap (y \sqcap (z \sqcup u)))) = x \sqcap (y \sqcap (z \sqcup u))$
using *13 26* **by** *metis*
have *60*: $\bigwedge x\ y\ .\ x \sqcup ((x \sqcap y) \sqcup (x \sqcap (y \sqcup top))) = x \sqcap (y \sqcup top)$
using *15 25 26* **by** *metis*
have *62*: $\bigwedge x\ y\ .\ x \sqcup ((x \sqcap -\ y) \sqcup (x \sqcap -\ -\ y)) = x$
using *9 15 19 25 26* **by** *metis*
have *65*: $\bigwedge x\ y\ .\ (-\ (x \sqcup y) \sqcap x) \sqcup (-\ (x \sqcup y) \sqcap y) = bot$
using *9 20 26 29* **by** *metis*
have *66*: $\bigwedge x\ y\ z\ .\ (x \sqcap -\ -\ y) \sqcup (x \sqcap -\ (-\ y \sqcap z)) = x \sqcap -\ (-\ y \sqcap z)$
using *11 24 26* **by** *metis*
have *69*: $\bigwedge x\ y\ .\ x \sqcup (x \sqcap -\ -\ y) = x$
using *9 15 26 30 50* **by** *metis*
have *81*: $\bigwedge x\ .\ top \sqcup -\ x = top$
using *9 19 30* **by** *metis*
have *82*: $\bigwedge x\ y\ z\ .\ (x \sqcap y) \sqcup (x \sqcap (y \sqcup z)) = x \sqcap (y \sqcup z)$
using *11 26 30* **by** *metis*
have *83*: $\bigwedge x\ y\ .\ x \sqcup (x \sqcap (y \sqcup top)) = x \sqcap (y \sqcup top)$
using *60 82* **by** *metis*
have *88*: $\bigwedge x\ y\ .\ x \sqcup (-\ y \sqcap x) = x$
using *14 17 81* **by** *metis*
have *89*: $\bigwedge x\ y\ .\ top \sqcup (x \sqcup -\ y) = x \sqcup top$
using *25 81* **by** *metis*
have *91*: $\bigwedge x\ y\ z\ .\ x \sqcup (y \sqcup (z \sqcup x)) = y \sqcup (z \sqcup x)$
using *8 32* **by** *metis*
have *94*: $\bigwedge x\ y\ z\ .\ x \sqcup (y \sqcup (-\ z \sqcap x)) = y \sqcup x$
using *25 88* **by** *metis*
have *101*: $\bigwedge x\ y\ z\ .\ x \sqcup (y \sqcup (x \sqcap -\ -\ z)) = y \sqcup x$
using *25 69* **by** *metis*
have *102*: $\bigwedge x\ .\ x \sqcup (x \sqcap bot) = x$
using *41 49 69* **by** *metis*

have *103*: $\bigwedge x\ y\ .\ x \sqcup (x \sqcap -\ y) = x$
using *9 62 101* **by** *metis*
have *109*: $\bigwedge x\ y\ .\ x \sqcup (y \sqcup (x \sqcap bot)) = y \sqcup x$
using *25 102* **by** *metis*
have *111*: $\bigwedge x\ y\ z\ .\ (-\ x \sqcap y) \sqcup ((-\ -\ x \sqcap y) \sqcup z) = y \sqcup z$
using *14 19 33* **by** *metis*
have *116*: $\bigwedge x\ y\ z\ .\ x \sqcup ((x \sqcap -\ y) \sqcup z) = x \sqcup z$
using *8 103* **by** *metis*
have *119*: $\bigwedge x\ y\ z\ .\ x \sqcup (y \sqcup (x \sqcap -\ z)) = y \sqcup x$
using *25 103* **by** *metis*
have *123*: $\bigwedge x\ .\ -\ -\ x \sqcap x = x$
using *14 19 42* **by** *metis*
have *127*: $\bigwedge x\ y\ .\ -\ -\ x \sqcap (x \sqcap y) = x \sqcap y$
using *13 123* **by** *metis*
have *130*: $\bigwedge x\ .\ -\ x \sqcup -\ -\ -\ x = -\ x$
using *9 24 123* **by** *metis*
have *132*: $\bigwedge x\ .\ -\ -\ -\ x = -\ x$
using *9 103 123 130* **by** *metis*
have *134*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ (-\ -\ x \sqcap y) = -\ (-\ -\ x \sqcap y)$
using *24 132* **by** *metis*
have *136*: $\bigwedge x\ y\ .\ (-\ x \sqcup y) \sqcap -\ -\ x = y \sqcap -\ -\ x$
using *42 132* **by** *metis*
have *138*: $\bigwedge x\ .\ -\ x \sqcap -\ x = -\ x$
using *123 132* **by** *metis*
have *144*: $\bigwedge x\ y\ z\ .\ ((-\ (x \sqcap y) \sqcap x) \sqcup z) \sqcap y = z \sqcap y$
using *20 29 34* **by** *metis*
have *157*: $\bigwedge x\ y\ .\ (-\ x \sqcup y) \sqcap -\ x = (top \sqcup y) \sqcap -\ x$
using *17 36 138* **by** *metis*
have *182*: $\bigwedge x\ y\ z\ .\ (x \sqcup (-\ -\ (y \sqcap z) \sqcap y)) \sqcap z = (x \sqcup y) \sqcap z$
using *17 35 123* **by** *metis*
have *288*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ (-\ x \sqcap y) = top$
using *24 38 81* **by** *metis*
have *315*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcap y) \sqcap x = x$
using *14 42 288* **by** *metis*
have *316*: $\bigwedge x\ y\ z\ .\ -\ (-\ x \sqcap y) \sqcap (x \sqcap z) = x \sqcap z$
using *13 315* **by** *metis*
have *319*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ -\ (-\ x \sqcap y) = -\ x$
using *9 24 315* **by** *metis*
have *320*: $\bigwedge x\ y\ .\ -\ -\ (-\ x \sqcap y) \sqcap x = bot$
using *40 315* **by** *metis*
have *373*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ (x \sqcap y) = -\ (x \sqcap y)$
using *24 127 132* **by** *metis*
have *387*: $\bigwedge x\ y\ .\ -\ (x \sqcap y) \sqcap -\ x = -\ x$
using *127 315* **by** *metis*
have *388*: $\bigwedge x\ y\ .\ -\ -\ (x \sqcap y) \sqcap -\ x = bot$
using *127 320* **by** *metis*
have *404*: $\bigwedge x\ y\ z\ .\ -\ (x \sqcap y) \sqcap (-\ x \sqcap z) = -\ x \sqcap z$
using *13 387* **by** *metis*
have *405*: $\bigwedge x\ y\ z\ .\ -\ (x \sqcap (y \sqcap z)) \sqcap -\ (x \sqcap y) = -\ (x \sqcap y)$

using *13 387* **by** *metis*
have *419*: $\bigwedge x\ y\ .\ -\ x \sqcap -\ -\ (-\ x \sqcap y) = -\ -\ (-\ x \sqcap y)$
using *315 387* **by** *metis*
have *420*: $\bigwedge x\ y\ .\ -\ -\ x \sqcap -\ -\ (x \sqcap y) = -\ -\ (x \sqcap y)$
using *387* **by** *metis*
have *421*: $\bigwedge x\ y\ z\ .\ -\ -\ (x \sqcap y) \sqcap (-\ x \sqcap z) = bot$
using *13 18 388* **by** *metis*
have *536*: $\bigwedge x\ y\ .\ (x \sqcup -\ -\ y) \sqcap y = (x \sqcup top) \sqcap y$
using *42 47* **by** *metis*
have *662*: $\bigwedge u\ x\ y\ z\ .\ (x \sqcap y) \sqcup ((x \sqcap (z \sqcup y)) \sqcup u) = (x \sqcap (z \sqcup y)) \sqcup u$
using *9 32 54* **by** *metis*
have *705*: $\bigwedge u\ x\ y\ z\ .\ (x \sqcap (y \sqcup z)) \sqcup ((x \sqcap (y \sqcup (z \sqcap bot))) \sqcup u) = (x \sqcap (y \sqcup z)) \sqcup u$
using *25 54 109 662* **by** *metis*
have *755*: $\bigwedge x\ y\ z\ .\ (x \sqcap -\ y) \sqcup (z \sqcup x) = z \sqcup x$
using *32 91 116* **by** *metis*
have *757*: $\bigwedge x\ y\ z\ .\ x \sqcup (x \sqcap (-\ y \sqcap -\ z)) = x$
using *13 103 116* **by** *metis*
have *930*: $\bigwedge x\ y\ z\ .\ (-\ (x \sqcap (y \sqcup z)) \sqcap (x \sqcap y)) \sqcup (-\ (x \sqcap (y \sqcup z)) \sqcap (x \sqcap z)) = bot$
using *9 20 29 58* **by** *metis*
have *1091*: $\bigwedge x\ y\ .\ -\ (x \sqcup y) \sqcap x = bot$
using *9 29 30 65* **by** *metis*
have *1092*: $\bigwedge x\ y\ .\ -\ (x \sqcup y) \sqcap y = bot$
using *29 30 65 1091* **by** *metis*
have *1113*: $\bigwedge u\ x\ y\ z\ .\ -\ (x \sqcup ((y \sqcup z) \sqcap u)) \sqcap (x \sqcup (z \sqcap u)) = bot$
using *29 46 65 1091* **by** *metis*
have *1117*: $\bigwedge x\ y\ z\ .\ -\ (x \sqcup y) \sqcap (x \sqcup (-\ z \sqcap y)) = bot$
using *29 65 94 1092* **by** *metis*
have *1128*: $\bigwedge x\ y\ z\ .\ -\ (x \sqcup (y \sqcup z)) \sqcap (x \sqcup y) = bot$
using *8 1091* **by** *metis*
have *1129*: $\bigwedge x\ y\ z\ .\ (-\ (x \sqcup y) \sqcup z) \sqcap x = z \sqcap x$
using *17 29 1091* **by** *metis*
have *1155*: $\bigwedge x\ y\ .\ -\ -\ x \sqcap -\ (-\ x \sqcap y) = -\ -\ x$
using *66 103 123 132* **by** *metis*
have *1578*: $\bigwedge x\ y\ z\ .\ -\ (x \sqcap (y \sqcup z)) \sqcap (x \sqcap y) = bot$
using *82 1091* **by** *metis*
have *1594*: $\bigwedge x\ y\ z\ .\ -\ (x \sqcap (y \sqcup z)) \sqcap (x \sqcap z) = bot$
using *29 930 1578* **by** *metis*
have *2094*: $\bigwedge x\ y\ z\ .\ -\ (x \sqcup (y \sqcap (z \sqcup top))) \sqcap (x \sqcup y) = bot$
using *83 1128* **by** *metis*
have *2097*: $\bigwedge x\ y\ .\ -\ -\ (x \sqcup y) \sqcap x = x$
using *14 19 1129* **by** *metis*
have *2124*: $\bigwedge x\ y\ .\ -\ -\ (x \sqcup y) \sqcap y = y$
using *9 2097* **by** *metis*
have *2135*: $\bigwedge x\ y\ .\ -\ -\ ((top \sqcup x) \sqcap y) \sqcap y = y$
using *36 2097* **by** *metis*
have *2136*: $\bigwedge x\ y\ .\ -\ -\ ((x \sqcup top) \sqcap y) \sqcap y = y$
using *37 2097* **by** *metis*

have *2137*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ -\ (x \sqcup y) = top$
using *9 288 2097* **by** *metis*
have *2138*: $\bigwedge x\ y\ .\ -\ x \sqcap -\ (x \sqcup y) = -\ (x \sqcup y)$
using *315 2097* **by** *metis*
have *2151*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ (x \sqcup y) = -\ x$
using *9 132 373 2097* **by** *metis*
have *2191*: $\bigwedge x\ y\ .\ -\ -\ x \sqcap (-\ y \sqcap x) = -\ y \sqcap x$
using *88 2124* **by** *metis*
have *2201*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ -\ (y \sqcup x) = top$
using *9 288 2124* **by** *metis*
have *2202*: $\bigwedge x\ y\ .\ -\ x \sqcap -\ (y \sqcup x) = -\ (y \sqcup x)$
using *315 2124* **by** *metis*
have *2320*: $\bigwedge x\ y\ .\ -\ (x \sqcap (y \sqcup top)) = -\ x$
using *83 373 2151* **by** *metis*
have *2343*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcap y) \sqcup -\ -\ y = top$
using *88 2201* **by** *metis*
have *2546*: $\bigwedge x\ y\ z\ .\ -\ x \sqcup ((-\ -\ x \sqcap -\ y) \sqcup z) = -\ x \sqcup (-\ y \sqcup z)$
using *111 116* **by** *metis*
have *2706*: $\bigwedge x\ y\ z\ .\ -\ x \sqcup (y \sqcup -\ -\ ((top \sqcup z) \sqcap -\ x)) = y \sqcup -\ -\ ((top \sqcup z) \sqcap -\ x)$
using *755 2135* **by** *metis*
have *2810*: $\bigwedge x\ y\ .\ -\ x \sqcap -\ ((y \sqcup top) \sqcap x) = -\ ((y \sqcup top) \sqcap x)$
using *315 2136* **by** *metis*
have *3022*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ (-\ y \sqcap -\ x) = top$
using *9 132 2343* **by** *metis*
have *3133*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcap -\ y) \sqcap y = y$
using *14 42 3022* **by** *metis*
have *3134*: $\bigwedge x\ y\ .\ -\ x \sqcap (-\ y \sqcap -\ x) = -\ y \sqcap -\ x$
using *14 43 3022* **by** *metis*
have *3961*: $\bigwedge x\ y\ .\ -\ -\ (x \sqcup y) \sqcap -\ -\ x = -\ -\ x$
using *14 136 2137* **by** *metis*
have *4644*: $\bigwedge x\ y\ z\ .\ -\ (x \sqcap -\ y) \sqcap (x \sqcap -\ (y \sqcup z)) = bot$
using *1594 2151* **by** *metis*
have *5495*: $\bigwedge x\ y\ z\ .\ -\ -\ (x \sqcap y) \sqcap -\ (x \sqcup z) = bot$
using *421 2138* **by** *metis*
have *9413*: $\bigwedge x\ y\ .\ -\ -\ (-\ x \sqcap y) \sqcap y = -\ x \sqcap y$
using *9 103 182 319* **by** *metis*
have *9519*: $\bigwedge x\ y\ z\ .\ -\ -\ (-\ x \sqcap y) \sqcap -\ -\ (x \sqcap z) = bot$
using *373 5495* **by** *metis*
have *11069*: $\bigwedge x\ y\ z\ .\ -\ (-\ -\ x \sqcap y) \sqcup (-\ x \sqcap -\ z) = -\ (-\ -\ x \sqcap y)$
using *316 757* **by** *metis*
have *12370*: $\bigwedge x\ y\ .\ -\ x \sqcap -\ (-\ -\ x \sqcap y) = -\ x$
using *132 1155* **by** *metis*
have *12376*: $\bigwedge x\ y\ .\ -\ x \sqcap -\ (x \sqcap y) = -\ x$
using *127 132 1155* **by** *metis*
have *12383*: $\bigwedge x\ y\ .\ -\ (x \sqcup y) \sqcap -\ y = -\ (x \sqcup y)$
using *132 1155 2124* **by** *metis*
have *12393*: $\bigwedge x\ y\ .\ -\ -\ (-\ x \sqcap -\ y) = -\ x \sqcap -\ y$
using *1155 3133 9413* **by** *metis*

have *12407*: $\bigwedge x\ y\ .\ -\ -\ x \sqcap -\ -\ (x \sqcup y) = -\ -\ x$
  using *1155 2138* **by** *metis*
have *12639*: $\bigwedge x\ y\ .\ -\ x \sqcap -\ (-\ y \sqcap x) = -\ x$
  using *88 12383* **by** *metis*
have *24647*: $\bigwedge x\ y\ .\ (-\ x \sqcap -\ y) \sqcup -\ (-\ x \sqcap -\ y) = top$
  using *19 12393* **by** *metis*
have *28269*: $\bigwedge x\ y\ z\ .\ -\ -\ (x \sqcap y) \sqcup -\ (-\ x \sqcap z) = -\ (-\ x \sqcap z)$
  using *373 404* **by** *metis*
have *28338*: $\bigwedge x\ y\ .\ -\ (-\ -\ (x \sqcap y) \sqcap x) = -\ (x \sqcap y)$
  using *123 405 12370* **by** *metis*
have *28422*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcap -\ y) = -\ (-\ y \sqcap -\ x)$
  using *13 3134 12393 28338* **by** *metis*
have *28485*: $\bigwedge x\ y\ .\ -\ x \sqcap -\ y = -\ y \sqcap -\ x$
  using *2097 3961 12393 28422* **by** *metis*
have *30411*: $\bigwedge x\ y\ .\ -\ x \sqcap (x \sqcup (x \sqcap y)) = bot$
  using *9 82 2094 2320* **by** *metis*
have *30469*: $\bigwedge x\ .\ -\ x \sqcap (x \sqcup -\ -\ x) = bot$
  using *9 123 132 30411* **by** *metis*
have *37513*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcap -\ y) \sqcap -\ (y \sqcup x) = bot$
  using *2202 4644* **by** *metis*
have *52421*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcap -\ (-\ x \sqcap y)) \sqcap y = y$
  using *14 144 24647 28485* **by** *metis*
have *52520*: $\bigwedge x\ y\ .\ -\ x \sqcap -\ (-\ x \sqcap y) = -\ x \sqcap -\ y$
  using *13 12376 12393 12639 28485 52421* **by** *metis*
have *52533*: $\bigwedge x\ y\ z\ .\ -\ -\ (x \sqcup (y \sqcap (z \sqcup top))) \sqcap (x \sqcup y) = x \sqcup y$
  using *15 49 2094 52421* **by** *metis*
have *61101*: $\bigwedge x\ y\ z\ .\ -\ (-\ -\ x \sqcap y) \sqcup z = -\ x \sqcup (-\ y \sqcup z)$
  using *111 2546 12370 52520* **by** *metis*
have *61156*: $\bigwedge x\ y\ .\ -\ -\ (-\ x \sqcap y) = -\ x \sqcap -\ -\ y$
  using *419 52520* **by** *metis*
have *61162*: $\bigwedge x\ y\ .\ -\ (x \sqcup (x \sqcap y)) = -\ x$
  using *15 49 2138 30411 52520* **by** *metis*
have *61163*: $\bigwedge x\ .\ -\ (x \sqcup -\ -\ x) = -\ x$
  using *15 49 2138 30469 52520* **by** *metis*
have *61229*: $\bigwedge x\ y\ z\ .\ -\ x \sqcap (-\ -\ y \sqcap -\ (x \sqcap z)) = -\ x \sqcap -\ -\ y$
  using *13 15 49 132 9519 52520 61156* **by** *metis*
have *61311*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ y = -\ (-\ -\ y \sqcap x)$
  using *119 11069 61101* **by** *metis*
have *61391*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcap -\ -\ y) = -\ (-\ x \sqcap y)$
  using *13 28269 61156 61229 61311* **by** *metis*
have *61420*: $\bigwedge x\ y\ .\ -\ (-\ -\ x \sqcap y) = -\ (-\ -\ y \sqcap x)$
  using *13 134 2191 61156 61311* **by** *metis*
have *61454*: $\bigwedge x\ y\ .\ -\ (x \sqcup -\ (-\ y \sqcap -\ x)) = -\ y \sqcap -\ x$
  using *9 132 3133 61156 61162* **by** *metis*
have *61648*: $\bigwedge x\ y\ .\ -\ x \sqcap (x \sqcup (-\ y \sqcap -\ -\ x)) = bot$
  using *1117 61163* **by** *metis*
have *62434*: $\bigwedge x\ y\ .\ -\ (-\ -\ x \sqcap y) \sqcap x = -\ y \sqcap x$
  using *43 61311* **by** *metis*
have *63947*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcap y) \sqcap -\ (-\ y \sqcup x) = bot$

using *37513 61391* **by** *metis*
have *64227*: $\bigwedge x\ y\ .\ -\ (x \sqcup (-\ y \sqcap -\ -\ x)) = -\ x$
using *15 49 2138 52520 61648* **by** *metis*
have *64239*: $\bigwedge x\ y\ .\ -\ (x \sqcup (-\ -\ x \sqcup y)) = -\ (x \sqcup y)$
using *9 25 12407 64227* **by** *metis*
have *64241*: $\bigwedge x\ y\ .\ -\ (x \sqcup (-\ -\ x \sqcap -\ y)) = -\ x$
using *28485 64227* **by** *metis*
have *64260*: $\bigwedge x\ y\ .\ -\ (x \sqcup -\ -\ (x \sqcap y)) = -\ x$
using *420 64241* **by** *metis*
have *64271*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcup (y \sqcup -\ -\ (y \sqcap x))) = -\ (-\ x \sqcup y)$
using *9 25 42 64260* **by** *metis*
have *64281*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcup y) = -\ (y \sqcup -\ -\ ((top \sqcup y) \sqcap -\ x))$
using *9 25 157 2706 64260* **by** *metis*
have *64282*: $\bigwedge x\ y\ .\ -\ (x \sqcup -\ -\ ((x \sqcup top) \sqcap y)) = -\ (x \sqcup -\ -\ y)$
using *9 25 132 536 2810 28485 61311 64260* **by** *metis*
have *65110*: $\bigwedge x\ y\ .\ -\ ((-\ x \sqcap y) \sqcup (-\ y \sqcup x)) = bot$
using *9 14 49 37513 63947* **by** *metis*
have *65231*: $\bigwedge x\ y\ .\ -\ (x \sqcup ((-\ x \sqcap y) \sqcup -\ y)) = bot$
using *9 25 65110* **by** *metis*
have *65585*: $\bigwedge x\ y\ .\ -\ (x \sqcup -\ y) = -\ -\ y \sqcap -\ x$
using *61311 61454 64239* **by** *metis*
have *65615*: $\bigwedge x\ y\ .\ -\ x \sqcap -\ ((x \sqcup top) \sqcap y) = -\ y \sqcap -\ x$
using *132 28485 64282 65585* **by** *metis*
have *65616*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcup y) = -\ y \sqcap -\ ((top \sqcup y) \sqcap -\ x)$
using *132 28485 64281 65585* **by** *metis*
have *65791*: $\bigwedge x\ y\ .\ -\ x \sqcap -\ ((top \sqcup x) \sqcap -\ y) = -\ -\ y \sqcap -\ x$
using *89 132 12376 28485 64271 65585 65615 65616* **by** *metis*
have *65933*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcup y) = -\ -\ x \sqcap -\ y$
using *65616 65791* **by** *metis*
have *66082*: $\bigwedge x\ y\ z\ .\ -\ (x \sqcup (y \sqcup -\ z)) = -\ -\ z \sqcap -\ (x \sqcup y)$
using *8 65585* **by** *metis*
have *66204*: $\bigwedge x\ y\ .\ -\ -\ x \sqcap -\ (y \sqcup (-\ y \sqcap x)) = bot$
using *65231 66082* **by** *metis*
have *66281*: $\bigwedge x\ y\ z\ .\ -\ (x \sqcup (-\ y \sqcup z)) = -\ -\ y \sqcap -\ (x \sqcup z)$
using *25 65933* **by** *metis*
have *67527*: $\bigwedge x\ y\ .\ -\ -\ (x \sqcup (-\ x \sqcap y)) \sqcap y = y$
using *14 49 62434 66204* **by** *metis*
have *67762*: $\bigwedge x\ y\ .\ -\ (-\ -\ x \sqcap (y \sqcup (-\ y \sqcap x))) = -\ x$
using *61420 67527* **by** *metis*
have *68018*: $\bigwedge x\ y\ z\ .\ -\ (x \sqcup y) \sqcap (x \sqcup (y \sqcap (z \sqcup top))) = bot$
using *8 83 1113 2320* **by** *metis*
have *71989*: $\bigwedge x\ y\ z\ .\ -\ (x \sqcup (y \sqcap (z \sqcup top))) = -\ (x \sqcup y)$
using *9 29 52533 67762 68018* **by** *metis*
have *71997*: $\bigwedge x\ y\ z\ .\ -\ ((x \sqcap (y \sqcup top)) \sqcup z) = -\ (x \sqcup z)$
using *17 2320 71989* **by** *metis*
have *72090*: $\bigwedge x\ y\ z\ .\ -\ (x \sqcup ((x \sqcap y) \sqcup z)) = -\ (x \sqcup z)$
using *10 14 705 71997* **by** *metis*
have *72139*: $\bigwedge x\ y\ .\ -\ (x \sqcup y) = -\ x \sqcap -\ y$
using *25 123 132 2138 65933 66281 72090* **by** *metis*

show *?thesis*
using *72139* **by** *metis*
qed

lemma *l15*:
$--(x \sqcup y) = --x \sqcup --y$
by (*simp add*: *l11 l12 l4*)

lemma *l13-var*:
$- - (- x \sqcap y) = - x \sqcap - - y$
proof −
have *1*: $\bigwedge x\ y\ .\ x \leq y \longleftrightarrow x \sqcup y = y$
by (*simp add*: *il-less-eq*)
have *4*: $\bigwedge x\ y\ .\ \neg(x \leq y) \vee x \sqcup y = y$
using *1* **by** *metis*
have *5*: $\bigwedge x\ y\ z\ .\ (x \sqcap y) \sqcup (x \sqcap z) \leq x \sqcap (y \sqcup z)$
by (*simp add*: *il-sub-inf-right-isotone-var*)
have *6*: $\bigwedge x\ y\ .\ - - x \leq - (- x \sqcap y)$
by (*simp add*: *pad2*)
have *7*: $\bigwedge x\ y\ z\ .\ x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$
by (*simp add*: *il-associative*)
have *8*: $\bigwedge x\ y\ z\ .\ (x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$
using *7* **by** *metis*
have *9*: $\bigwedge x\ y\ .\ x \sqcup y = y \sqcup x$
by (*simp add*: *il-commutative*)
have *10*: $\bigwedge x\ .\ x \sqcup bot = x$
by (*simp add*: *il-bot-unit*)
have *11*: $\bigwedge x\ .\ x \sqcup x = x$
by *simp*
have *12*: $\bigwedge x\ y\ z\ .\ x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$
by (*simp add*: *il-inf-associative*)
have *13*: $\bigwedge x\ y\ z\ .\ (x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$
using *12* **by** *metis*
have *14*: $\bigwedge x\ .\ top \sqcap x = x$
by *simp*
have *15*: $\bigwedge x\ .\ x \sqcap top = x$
by *simp*
have *16*: $\bigwedge x\ y\ z\ .\ (x \sqcup y) \sqcap z = (x \sqcap z) \sqcup (y \sqcap z)$
by (*simp add*: *il-inf-right-dist-sup*)
have *17*: $\bigwedge x\ y\ z\ .\ (x \sqcap y) \sqcup (z \sqcap y) = (x \sqcup z) \sqcap y$
using *16* **by** *metis*
have *19*: $\bigwedge x\ .\ - x \sqcup - - x = top$
by *simp*
have *20*: $\bigwedge x\ .\ - x \sqcap x = bot$
by (*simp add*: *a-inf-complement-bot*)
have *22*: $\bigwedge x\ y\ z\ .\ ((x \sqcap y) \sqcup (x \sqcap z)) \sqcup (x \sqcap (y \sqcup z)) = x \sqcap (y \sqcup z)$
using *4 5* **by** *metis*
have *23*: $\bigwedge x\ y\ z\ .\ (x \sqcap (y \sqcup z)) \sqcup ((x \sqcap y) \sqcup (x \sqcap z)) = x \sqcap (y \sqcup z)$
using *9 22* **by** *metis*

have *24*: $\bigwedge x\ y\ .\ -\ -\ x \sqcup -\ (-\ x \sqcap y) = -\ (-\ x \sqcap y)$
using *4 6* **by** *metis*
have *25*: $\bigwedge x\ y\ z\ .\ x \sqcup (y \sqcup z) = y \sqcup (x \sqcup z)$
using *8 9* **by** *metis*
have *26*: $\bigwedge x\ y\ z\ .\ (x \sqcap y) \sqcup ((x \sqcap z) \sqcup (x \sqcap (y \sqcup z))) = x \sqcap (y \sqcup z)$
using *9 23 25* **by** *metis*
have *29*: $\bigwedge x\ .\ bot \sqcup x = x$
using *9 10* **by** *metis*
have *30*: $\bigwedge x\ y\ .\ x \sqcup (x \sqcup y) = x \sqcup y$
using *8 11* **by** *metis*
have *34*: $\bigwedge u\ x\ y\ z\ .\ (x \sqcap (y \sqcap z)) \sqcup (u \sqcap z) = ((x \sqcap y) \sqcup u) \sqcap z$
using *13 17* **by** *metis*
have *35*: $\bigwedge u\ x\ y\ z\ .\ (x \sqcap y) \sqcup (z \sqcap (u \sqcap y)) = (x \sqcup (z \sqcap u)) \sqcap y$
using *13 17* **by** *metis*
have *38*: $\bigwedge x\ y\ .\ -\ x \sqcup (-\ -\ x \sqcup y) = top \sqcup y$
using *8 19* **by** *metis*
have *41*: $-\ top = bot$
using *15 20* **by** *metis*
have *42*: $\bigwedge x\ y\ .\ (-\ x \sqcup y) \sqcap x = y \sqcap x$
using *17 20 29* **by** *metis*
have *43*: $\bigwedge x\ y\ .\ (x \sqcup -\ y) \sqcap y = x \sqcap y$
using *9 17 20 29* **by** *metis*
have *45*: $\bigwedge x\ .\ -\ bot \sqcup -\ -\ x = -\ bot$
using *9 20 24* **by** *metis*
have *49*: $-\ bot = top$
using *19 29 41* **by** *metis*
have *50*: $\bigwedge x\ .\ top \sqcup -\ -\ x = top$
using *45 49* **by** *metis*
have *62*: $\bigwedge x\ y\ .\ x \sqcup ((x \sqcap -\ y) \sqcup (x \sqcap -\ -\ y)) = x$
using *9 15 19 25 26* **by** *metis*
have *65*: $\bigwedge x\ y\ .\ (-\ (x \sqcup y) \sqcap x) \sqcup (-\ (x \sqcup y) \sqcap y) = bot$
using *9 20 26 29* **by** *metis*
have *66*: $\bigwedge x\ y\ z\ .\ (x \sqcap -\ -\ y) \sqcup (x \sqcap -\ (-\ y \sqcap z)) = x \sqcap -\ (-\ y \sqcap z)$
using *11 24 26* **by** *metis*
have *69*: $\bigwedge x\ y\ .\ x \sqcup (x \sqcap -\ -\ y) = x$
using *9 15 26 30 50* **by** *metis*
have *81*: $\bigwedge x\ .\ top \sqcup -\ x = top$
using *9 19 30* **by** *metis*
have *88*: $\bigwedge x\ y\ .\ x \sqcup (-\ y \sqcap x) = x$
using *14 17 81* **by** *metis*
have *101*: $\bigwedge x\ y\ z\ .\ x \sqcup (y \sqcup (x \sqcap -\ -\ z)) = y \sqcup x$
using *25 69* **by** *metis*
have *103*: $\bigwedge x\ y\ .\ x \sqcup (x \sqcap -\ y) = x$
using *9 62 101* **by** *metis*
have *123*: $\bigwedge x\ .\ -\ -\ x \sqcap x = x$
using *14 19 42* **by** *metis*
have *127*: $\bigwedge x\ y\ .\ -\ -\ x \sqcap (x \sqcap y) = x \sqcap y$
using *13 123* **by** *metis*
have *130*: $\bigwedge x\ .\ -\ x \sqcup -\ -\ -\ x = -\ x$

using *9 24 123* **by** *metis*
have *132*: $\bigwedge x\ .\ -\ -\ -\ x = -\ x$
using *9 103 123 130* **by** *metis*
have *136*: $\bigwedge x\ y\ .\ (-\ x \sqcup y) \sqcap -\ -\ x = y \sqcap -\ -\ x$
using *42 132* **by** *metis*
have *144*: $\bigwedge x\ y\ z\ .\ ((-\ (x \sqcap y) \sqcap x) \sqcup z) \sqcap y = z \sqcap y$
using *20 29 34* **by** *metis*
have *182*: $\bigwedge x\ y\ z\ .\ (x \sqcup (-\ -\ (y \sqcap z) \sqcap y)) \sqcap z = (x \sqcup y) \sqcap z$
using *17 35 123* **by** *metis*
have *288*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ (-\ x \sqcap y) = top$
using *24 38 81* **by** *metis*
have *315*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcap y) \sqcap x = x$
using *14 42 288* **by** *metis*
have *319*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ -\ (-\ x \sqcap y) = -\ x$
using *9 24 315* **by** *metis*
have *387*: $\bigwedge x\ y\ .\ -\ (x \sqcap y) \sqcap -\ x = -\ x$
using *127 315* **by** *metis*
have *405*: $\bigwedge x\ y\ z\ .\ -\ (x \sqcap (y \sqcap z)) \sqcap -\ (x \sqcap y) = -\ (x \sqcap y)$
using *13 387* **by** *metis*
have *419*: $\bigwedge x\ y\ .\ -\ x \sqcap -\ -\ (-\ x \sqcap y) = -\ -\ (-\ x \sqcap y)$
using *315 387* **by** *metis*
have *1091*: $\bigwedge x\ y\ .\ -\ (x \sqcup y) \sqcap x = bot$
using *9 29 30 65* **by** *metis*
have *1129*: $\bigwedge x\ y\ z\ .\ (-\ (x \sqcup y) \sqcup z) \sqcap x = z \sqcap x$
using *17 29 1091* **by** *metis*
have *1155*: $\bigwedge x\ y\ .\ -\ -\ x \sqcap -\ (-\ x \sqcap y) = -\ -\ x$
using *66 103 123 132* **by** *metis*
have *2097*: $\bigwedge x\ y\ .\ -\ -\ (x \sqcup y) \sqcap x = x$
using *14 19 1129* **by** *metis*
have *2124*: $\bigwedge x\ y\ .\ -\ -\ (x \sqcup y) \sqcap y = y$
using *9 2097* **by** *metis*
have *2137*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ -\ (x \sqcup y) = top$
using *9 288 2097* **by** *metis*
have *2201*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ -\ (y \sqcup x) = top$
using *9 288 2124* **by** *metis*
have *2343*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcap y) \sqcup -\ -\ y = top$
using *88 2201* **by** *metis*
have *3022*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ (-\ y \sqcap -\ x) = top$
using *9 132 2343* **by** *metis*
have *3133*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcap -\ y) \sqcap y = y$
using *14 42 3022* **by** *metis*
have *3134*: $\bigwedge x\ y\ .\ -\ x \sqcap (-\ y \sqcap -\ x) = -\ y \sqcap -\ x$
using *14 43 3022* **by** *metis*
have *3961*: $\bigwedge x\ y\ .\ -\ -\ (x \sqcup y) \sqcap -\ -\ x = -\ -\ x$
using *14 136 2137* **by** *metis*
have *9413*: $\bigwedge x\ y\ .\ -\ -\ (-\ x \sqcap y) \sqcap y = -\ x \sqcap y$
using *9 103 182 319* **by** *metis*
have *12370*: $\bigwedge x\ y\ .\ -\ x \sqcap -\ (-\ -\ x \sqcap y) = -\ x$
using *132 1155* **by** *metis*

have *12376*: $\bigwedge x\ y\ .\ -\ x \sqcap -\ (x \sqcap y) = -\ x$
  using *127 132 1155* **by** *metis*
have *12383*: $\bigwedge x\ y\ .\ -\ (x \sqcup y) \sqcap -\ y = -\ (x \sqcup y)$
  using *132 1155 2124* **by** *metis*
have *12393*: $\bigwedge x\ y\ .\ -\ -\ (-\ x \sqcap -\ y) = -\ x \sqcap -\ y$
  using *1155 3133 9413* **by** *metis*
have *12639*: $\bigwedge x\ y\ .\ -\ x \sqcap -\ (-\ y \sqcap x) = -\ x$
  using *88 12383* **by** *metis*
have *24647*: $\bigwedge x\ y\ .\ (-\ x \sqcap -\ y) \sqcup -\ (-\ x \sqcap -\ y) = top$
  using *19 12393* **by** *metis*
have *28338*: $\bigwedge x\ y\ .\ -\ (-\ -\ (x \sqcap y) \sqcap x) = -\ (x \sqcap y)$
  using *123 405 12370* **by** *metis*
have *28422*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcap -\ y) = -\ (-\ y \sqcap -\ x)$
  using *13 3134 12393 28338* **by** *metis*
have *28485*: $\bigwedge x\ y\ .\ -\ x \sqcap -\ y = -\ y \sqcap -\ x$
  using *2097 3961 12393 28422* **by** *metis*
have *52421*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcap -\ (-\ x \sqcap y)) \sqcap y = y$
  using *14 144 24647 28485* **by** *metis*
have *52520*: $\bigwedge x\ y\ .\ -\ x \sqcap -\ (-\ x \sqcap y) = -\ x \sqcap -\ y$
  using *13 12376 12393 12639 28485 52421* **by** *metis*
have *61156*: $\bigwedge x\ y\ .\ -\ -\ (-\ x \sqcap y) = -\ x \sqcap -\ -\ y$
  using *419 52520* **by** *metis*
show *?thesis*
  using *61156* **by** *metis*
qed

Theorem 25.1

subclass *subset-boolean-algebra-2*
proof
show $\bigwedge x\ y\ z.\ x \sqcup (y \sqcup z) = x \sqcup y \sqcup z$
  by (*simp add*: *il-associative*)
show $\bigwedge x\ y.\ x \sqcup y = y \sqcup x$
  by (*simp add*: *il-commutative*)
show $\bigwedge x.\ x \sqcup x = x$
  by *simp*
show $\bigwedge x\ y.\ x \sqcup -\ (y \sqcup -\ y) = x$
  using *il-bot-unit l12 l6* **by** *auto*
show $\bigwedge x\ y.\ -\ (x \sqcup y) = -\ (-\ -\ x \sqcup -\ -\ y)$
  by (*metis l15 l4*)
show $\bigwedge x\ y.\ -\ x \sqcup -\ (-\ x \sqcup y) = -\ x \sqcup -\ y$
  by (*smt l11 l15 il-inf-right-dist-sup il-unit-bot l6 l7*)
qed

lemma *aa-test*:
$p = --p \Longrightarrow test\ p$
by (*metis ppa-ppd.d-closed*)

lemma *test-aa-increasing*:
$test\ p \Longrightarrow p \leq --p$

by (*simp add*: *ppa-ppd.d-increasing-sub-identity test-sub-identity*)

lemma *test* $p \Longrightarrow - - (p \sqcap x) \leq p$ **nitpick** [*expect=genuine*] **oops**
lemma *test* $p \Longrightarrow --p \leq p$ **nitpick** [*expect=genuine*] **oops**

end

class *pa-algebra* = *pa-semiring* + *minus* +
assumes *pa-minus-def*: $-x - -y = -(--x \sqcup -y)$
begin

subclass *subset-boolean-algebra-2-extended*
proof
show $bot = (THE\ x.\ \forall z.\ x = - (z \sqcup - z))$
using *l12 l6* **by** *auto*
thus $top = - (THE\ x.\ \forall z.\ x = - (z \sqcup - z))$
using *l2* **by** *blast*
show $\bigwedge x\ y.\ - x \sqcap - y = - (- - x \sqcup - - y)$
by (*metis l12 l4*)
show $\bigwedge x\ y.\ - x - - y = - (- - x \sqcup - y)$
by (*simp add*: *pa-minus-def*)
show $\bigwedge x\ y.\ (x \leq y) = (x \sqcup y = y)$
by (*simp add*: *il-less-eq*)
show $\bigwedge x\ y.\ (x < y) = (x \sqcup y = y \wedge y \sqcup x \neq x)$
by (*simp add*: *il-less-eq less-le-not-le*)
qed

lemma $\bigwedge x\ y.\ - (x \sqcap - - y) = - (x \sqcap y)$ **nitpick** [*expect=genuine*] **oops**

end

8.3 Antidomain Semirings

Definition 24

class *a-semiring* = *ppa-semiring* +
assumes *ad3*: $-(x \sqcap y) \leq -(x \sqcap --y)$
begin

lemma *l16*:
$- - x \leq - (- x \sqcap y)$
proof −
have *1*: $\bigwedge x\ y\ .\ x \leq y \longleftrightarrow x \sqcup y = y$
by (*simp add*: *il-less-eq*)
have *3*: $\bigwedge x\ y\ z\ .\ x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$
by (*simp add*: *il-associative*)
have *4*: $\bigwedge x\ y\ z\ .\ (x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$
using *3* **by** *metis*
have *5*: $\bigwedge x\ y\ .\ x \sqcup y = y \sqcup x$
by (*simp add*: *il-commutative*)

have *6*: $\bigwedge x \,.\, x \sqcup bot = x$
  by (*simp add*: *il-bot-unit*)
have *7*: $\bigwedge x \,.\, x \sqcup x = x$
  by *simp*
have *8*: $\bigwedge x\ y \,.\, \neg(x \leq y) \vee x \sqcup y = y$
  using *1* **by** *metis*
have *9*: $\bigwedge x\ y \,.\, x \leq y \vee x \sqcup y \neq y$
  using *1* **by** *metis*
have *10*: $\bigwedge x\ y\ z \,.\, x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$
  by (*simp add*: *il-inf-associative*)
have *11*: $\bigwedge x\ y\ z \,.\, (x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$
  using *10* **by** *metis*
have *12*: $\bigwedge x \,.\, top \sqcap x = x$
  by *simp*
have *13*: $\bigwedge x \,.\, x \sqcap top = x$
  by *simp*
have *14*: $\bigwedge x\ y\ z \,.\, (x \sqcap y) \sqcup (x \sqcap z) \leq x \sqcap (y \sqcup z)$
  by (*simp add*: *il-sub-inf-right-isotone-var*)
have *15*: $\bigwedge x\ y\ z \,.\, (x \sqcup y) \sqcap z = (x \sqcap z) \sqcup (y \sqcap z)$
  by (*simp add*: *il-inf-right-dist-sup*)
have *16*: $\bigwedge x\ y\ z \,.\, (x \sqcap y) \sqcup (z \sqcap y) = (x \sqcup z) \sqcap y$
  using *15* **by** *metis*
have *17*: $\bigwedge x \,.\, bot \sqcap x = bot$
  by *simp*
have *18*: $\bigwedge x \,.\, -\ x \sqcup -\ -\ x = top$
  by *simp*
have *19*: $\bigwedge x \,.\, -\ x \sqcap x = bot$
  by (*simp add*: *a-inf-complement-bot*)
have *20*: $\bigwedge x\ y \,.\, -\ (x \sqcap y) \leq -\ (x \sqcap -\ -\ y)$
  by (*simp add*: *ad3*)
have *22*: $\bigwedge x\ y\ z \,.\, x \sqcup (y \sqcup z) = y \sqcup (x \sqcup z)$
  using *4 5* **by** *metis*
have *25*: $\bigwedge x \,.\, bot \sqcup x = x$
  using *5 6* **by** *metis*
have *26*: $\bigwedge x\ y \,.\, x \sqcup (x \sqcup y) = x \sqcup y$
  using *4 7* **by** *metis*
have *33*: $\bigwedge x\ y\ z \,.\, (x \sqcap y) \sqcup ((x \sqcap z) \sqcup (x \sqcap (y \sqcup z))) = x \sqcap (y \sqcup z)$
  using *5 8 14 22* **by** *metis*
have *47*: $\bigwedge x\ y \,.\, -\ x \sqcup (-\ -\ x \sqcup y) = top \sqcup y$
  using *4 18* **by** *metis*
have *48*: $\bigwedge x\ y \,.\, -\ -\ x \sqcup (y \sqcup -\ x) = y \sqcup top$
  using *4 5 18* **by** *metis*
have *51*: $\bigwedge x\ y \,.\, -\ x \sqcap (x \sqcap y) = bot$
  using *11 17 19* **by** *metis*
have *52*: $-\ top = bot$
  using *13 19* **by** *metis*
have *56*: $\bigwedge x\ y \,.\, (-\ x \sqcup y) \sqcap x = y \sqcap x$
  using *16 19 25* **by** *metis*
have *57*: $\bigwedge x\ y \,.\, (x \sqcup -\ y) \sqcap y = x \sqcap y$

using *5 16 19 25* **by** *metis*
have *58*: $\bigwedge x\ y\ .\ -\ (x \sqcap y) \sqcup -\ (x \sqcap -\ -\ y) = -\ (x \sqcap -\ -\ y)$
using *8 20* **by** *metis*
have *60*: $\bigwedge x\ .\ -\ x \leq -\ -\ -\ x$
using *12 20* **by** *metis*
have *69*: $-\ bot = top$
using *18 25 52* **by** *metis*
have *74*: $\bigwedge x\ y\ .\ x \leq x \sqcup y$
using *9 26* **by** *metis*
have *78*: $\bigwedge x\ .\ top \sqcup -\ x = top$
using *5 18 26* **by** *metis*
have *80*: $\bigwedge x\ y\ .\ x \leq y \sqcup x$
using *5 74* **by** *metis*
have *86*: $\bigwedge x\ y\ z\ .\ x \sqcup y \leq x \sqcup (z \sqcup y)$
using *22 80* **by** *metis*
have *95*: $\bigwedge x\ .\ -\ x \sqcup -\ -\ -\ x = -\ -\ -\ x$
using *8 60* **by** *metis*
have *143*: $\bigwedge x\ y\ .\ x \sqcup (x \sqcap -\ y) = x$
using *5 13 26 33 78* **by** *metis*
have *370*: $\bigwedge x\ y\ z\ .\ x \sqcup (y \sqcap -\ z) \leq x \sqcup y$
using *86 143* **by** *metis*
have *907*: $\bigwedge x\ .\ -\ x \sqcap -\ x = -\ x$
using *12 18 57* **by** *metis*
have *928*: $\bigwedge x\ y\ .\ -\ x \sqcap (-\ x \sqcap y) = -\ x \sqcap y$
using *11 907* **by** *metis*
have *966*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcap -\ -\ (x \sqcap y)) = top$
using *51 58 69 78* **by** *metis*
have *1535*: $\bigwedge x\ .\ -\ x \sqcup -\ -\ -\ -\ x = top$
using *47 78 95* **by** *metis*
have *1630*: $\bigwedge x\ y\ z\ .\ (x \sqcup y) \sqcap -\ z \leq (x \sqcap -\ z) \sqcup y$
using *16 370* **by** *metis*
have *2422*: $\bigwedge x\ .\ -\ x \sqcap -\ -\ -\ x = -\ -\ -\ x$
using *12 57 1535* **by** *metis*
have *6567*: $\bigwedge x\ y\ .\ -\ x \sqcap -\ -\ (x \sqcap y) = bot$
using *12 19 966* **by** *metis*
have *18123*: $\bigwedge x\ .\ -\ -\ -\ x = -\ x$
using *95 143 2422* **by** *metis*
have *26264*: $\bigwedge x\ y\ .\ -\ x \leq (-\ y \sqcap -\ x) \sqcup -\ -\ y$
using *12 18 1630* **by** *metis*
have *26279*: $\bigwedge x\ y\ .\ -\ -\ (x \sqcap y) \leq -\ -\ x$
using *25 6567 26264* **by** *metis*
have *26307*: $\bigwedge x\ y\ .\ -\ -\ (-\ x \sqcap y) \leq -\ x$
using *928 18123 26279* **by** *metis*
have *26339*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ -\ (-\ x \sqcap y) = -\ x$
using *5 8 26307* **by** *metis*
have *26564*: $\bigwedge x\ y\ .\ -\ x \sqcup -\ (-\ x \sqcap y) = top$
using *5 48 78 18123 26339* **by** *metis*
have *26682*: $\bigwedge x\ y\ .\ -\ (-\ x \sqcap y) \sqcap x = x$
using *12 56 26564* **by** *metis*

have *26864*: $\bigwedge x\ y\ .\ -\ -\ x \leq -\ (-\ x \sqcap y)$
using *18123 26279 26682* **by** *metis*
show *?thesis*
using *26864* **by** *metis*
qed

Theorem 25.2

subclass *pa-semiring*
proof
show $\bigwedge x\ y.\ -\ -\ x \leq -\ (-\ x \sqcap y)$
by (*rule l16*)
qed

lemma *l17*:
$-(x \sqcap y) = -(x \sqcap --y)$
by (*simp add*: *ad3 antisym l14*)

lemma *a-complement-inf-double-complement*:
$-(x \sqcap --y) = -(x \sqcap y)$
using *l17* **by** *auto*

sublocale *a-d*: *d-semiring-var* **where** $d = \lambda x\ .\ --x$
proof
show $\bigwedge x\ y.\ -\ -\ (x \sqcap -\ -\ y) \leq -\ -\ (x \sqcap y)$
using *l17* **by** *auto*
show $-\ -\ bot = bot$
by (*simp add*: *l1 l2*)
qed

lemma *test* $p \Longrightarrow -\ -\ (p \sqcap x) \leq p$
by (*fact a-d.d2*)

end

class *a-algebra* = *a-semiring* + *minus* +
assumes *a-minus-def*: $-x - -y = -(--x \sqcup -y)$
begin

subclass *pa-algebra*
proof
show $\bigwedge x\ y.\ -\ x\ -\ -\ y = -\ (-\ -\ x \sqcup -\ y)$
by (*simp add*: *a-minus-def*)
qed

Theorem 25.4

subclass *subset-boolean-algebra-4-extended*
proof
show $\bigwedge x\ y\ z.\ x \sqcap (y \sqcap z) = x \sqcap y \sqcap z$
by (*simp add*: *il-inf-associative*)

show $\bigwedge x\ y\ z.\ (x \sqcup y) \sqcap z = x \sqcap z \sqcup y \sqcap z$
by (*simp add*: *il-inf-right-dist-sup*)
show $\bigwedge x.\ -\ x \sqcap x = bot$
by (*simp add*: *a-inf-complement-bot*)
show $\bigwedge x.\ top \sqcap x = x$
by *simp*
show $\bigwedge x\ y.\ -\ (x \sqcap -\ -\ y) = -\ (x \sqcap y)$
using *l17* **by** *auto*
show $\bigwedge x.\ x \sqcap top = x$
by *simp*
show $\bigwedge x\ y\ z.\ x \leq y \Longrightarrow z \sqcap x \leq z \sqcap y$
by (*simp add*: *il-sub-inf-right-isotone*)
qed

end

context *subset-boolean-algebra-4-extended*
begin

subclass *il-semiring*
proof
show $\bigwedge x\ y\ z.\ x \sqcup (y \sqcup z) = x \sqcup y \sqcup z$
by (*simp add*: *sup-assoc*)
show $\bigwedge x\ y.\ x \sqcup y = y \sqcup x$
by (*simp add*: *sup-commute*)
show $\bigwedge x.\ x \sqcup x = x$
by *simp*
show $\bigwedge x.\ x \sqcup bot = x$
by *simp*
show $\bigwedge x\ y\ z.\ x \sqcap (y \sqcap z) = x \sqcap y \sqcap z$
by (*simp add*: *sba3-inf-associative*)
show $\bigwedge x\ y\ z.\ (x \sqcup y) \sqcap z = x \sqcap z \sqcup y \sqcap z$
by (*simp add*: *sba3-inf-right-dist-sup*)
show $\bigwedge x.\ top \sqcap x = x$
by *simp*
show $\bigwedge x.\ x \sqcap top = x$
by *simp*
show $\bigwedge x.\ bot \sqcap x = bot$
by (*simp add*: *inf-left-zero*)
show $\bigwedge x\ y\ z.\ x \leq y \Longrightarrow z \sqcap x \leq z \sqcap y$
by (*simp add*: *inf-right-isotone*)
show $\bigwedge x\ y.\ (x \leq y) = (x \sqcup y = y)$
by (*simp add*: *le-iff-sup*)
show $\bigwedge x\ y.\ (x < y) = (x \leq y \wedge \neg\ y \leq x)$
by (*simp add*: *less-le-not-le*)
qed

subclass *a-semiring*
proof

show $\bigwedge x. - x \sqcap x = bot$
by (*simp add*: *sba3-inf-complement-bot*)
show $\bigwedge x. - x \sqcup - - x = top$
by *simp*
show $\bigwedge x\ y. - (x \sqcap y) \le - (x \sqcap - - y)$
by (*simp add*: *sba3-complement-inf-double-complement*)
qed

sublocale *sba4-a*: *a-algebra*
proof
show $\bigwedge x\ y. - x - - y = - (- - x \sqcup - y)$
by (*simp add*: *sub-minus-def*)
qed

end

context *stone-algebra*
begin

Theorem 25.3

subclass *il-semiring*
proof
show $\bigwedge x\ y\ z.\ x \sqcup (y \sqcup z) = x \sqcup y \sqcup z$
by (*simp add*: *sup-assoc*)
show $\bigwedge x\ y.\ x \sqcup y = y \sqcup x$
by (*simp add*: *sup-commute*)
show $\bigwedge x.\ x \sqcup x = x$
by *simp*
show $\bigwedge x.\ x \sqcup bot = x$
by *simp*
show $\bigwedge x\ y\ z.\ x \sqcap (y \sqcap z) = x \sqcap y \sqcap z$
by (*simp add*: *inf.sup-monoid.add-assoc*)
show $\bigwedge x\ y\ z.\ (x \sqcup y) \sqcap z = x \sqcap z \sqcup y \sqcap z$
by (*simp add*: *inf-sup-distrib2*)
show $\bigwedge x.\ top \sqcap x = x$
by *simp*
show $\bigwedge x.\ x \sqcap top = x$
by *simp*
show $\bigwedge x.\ bot \sqcap x = bot$
by *simp*
show $\bigwedge x\ y\ z.\ x \le y \Longrightarrow z \sqcap x \le z \sqcap y$
using *inf.sup-right-isotone* **by** *blast*
show $\bigwedge x\ y.\ (x \le y) = (x \sqcup y = y)$
by (*simp add*: *le-iff-sup*)
show $\bigwedge x\ y.\ (x < y) = (x \le y \wedge \neg\ y \le x)$
by (*simp add*: *less-le-not-le*)
qed

subclass *a-semiring*

proof
  show $\bigwedge x. - x \sqcap x = bot$
    by *simp*
  show $\bigwedge x. - x \sqcup - - x = top$
    by *simp*
  show $\bigwedge x\ y. - (x \sqcap y) \leq - (x \sqcap - - y)$
    by *simp*
qed

end

end

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