Closure, Properties and Closure Properties of Multirelations

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Abstract. Multirelations have been used for modelling games, protocols and computations. They have also been used for modelling contact, closure and topology. We bring together these two lines of research using relation algebras and more general algebras. In particular, we look at various properties of multirelations that have been used in the two lines of research, show how these properties are connected and study by which multirelational operations they are preserved. We find that many results do not require a restriction to up-closed multirelations; this includes connections between various kinds of reflexive-transitive closure.

1 Introduction

A multirelation is a relation between a set and a powerset. The powerset structure facilitates the modelling of two-player games or interaction between agents in a computation; see [5,17,19], for example. Already before these applications multirelations were used by G. Aumann to model contact and, thereby, to give beginners a more suggestive access to topology than traditional approaches do; see [1]. Properties of multirelations have been rediscovered over time, but, in our opinion, a systematic investigation is missing. The aim of the present paper is to start this research. Its methods are algebraic, in particular relation-algebraic.

The starting point is a relation-algebraic representation of multirelations and multirelational operations (Sections 2 and 3). Properties of these operations are proved using relation algebras and captured as axioms of more general structures based on lattices and semirings (Section 4). A key decision is to not specialise to up-closed multirelations at the outset, but to treat being up-closed as one among many properties a multirelation might have. This makes it possible to generalise results, for example, about closure operations (Section 5). Other properties are taken from the literature and compared systematically (Section 6). A particular question is whether they are preserved by multirelational operations (Section 7). Positive results are shown algebraically using Isabelle and automated theorem provers. Counterexamples are produced by a Haskell program. Moreover properties of topological contacts are derived from logical specifications (Section 8).

The contributions of the paper are (1) new algebraic structures, which capture (not only up-closed) multirelations, (2) a comparison of three reflexive-transitive closure operations in these algebras, (3) a study of relationships between properties of multirelations and (4) a study of preservation of these properties by multirelational operations. Overall, this paper brings together the topological and computational lines of research on multirelations. The companion paper [7] investigates how properties from these two lines of research translate to predicate transformers. It uses relation algebras to express the correspondence of multirelations and predicate transformers, which turns out to be similar to the correspondence between contact relations and closure operations.

2 Relation-Algebraic Prerequisites

In this section we present the facts on relations and heterogeneous relation algebras that are needed in the remainder of this paper. For more details on relations and relation algebras, see [25], for example.

We write $R : A \leftrightarrow B$ if R is a (typed binary) relation with source A and target B, that is, of type $A \leftrightarrow B$. If the sets A and B are finite, we may consider R as a Boolean matrix. Since this interpretation is well suited for many purposes, we will use matrix notation and write $R_{x,y}$ instead of $(x, y) \in R$ or $x\,R\,y$.

We assume the reader to be familiar with the basic operations on relations, namely R^{c} (converse), $\overline{R}$ (complement), $R \cup S$ (union), $R \cap S$ (intersection), RS (composition), the predicates indicating $R \subseteq S$ (inclusion) and $R = S$ (equality) and the special relations O (empty relation), T (universal relation) and I (identity relation). Converse has higher precedence than composition, which has higher precedence than union and intersection. The set of all relations of type $A \leftrightarrow B$ with the operations $\overline{}$, $\cup$, $\cap$, the ordering $\subseteq$ and the constants O and T forms a complete Boolean lattice. Further well-known rules are, for example, $(R^{\mathsf{c}})^{\mathsf{c}} = R$, $\overline{R^{\mathsf{c}}} = \overline{R}^{\mathsf{c}}$, and that $R \subseteq S$ implies $R^{\mathsf{c}} \subseteq S^{\mathsf{c}}$ as well as $RP \subseteq SP$ and $QR \subseteq QS$, for all P, Q, R and S.

The theoretical framework for these rules and many others is that of a (heterogeneous) relation algebra; see [27] for details. As constants and operations of this algebraic structure we have those of concrete (that is, set-theoretic) relations. The axioms of a relation algebra are those of a complete Boolean lattice for the Boolean part, the associativity and neutrality of identity relations for composition, the equivalence of $QR \subseteq S$, $Q^{\mathsf{c}}\overline{S} \subseteq \overline{R}$ and $\overline{S}R^{\mathsf{c}} \subseteq \overline{Q}$, for all relations Q, R and S – called the *Schröder equivalences* – and that $R \neq \mathsf{O}$ implies $\mathsf{T}R\mathsf{T} = \mathsf{T}$, for all relations R.

Residuals are the greatest solutions of certain inclusions. The *left residual* of S over R, in symbols S/R, is the greatest relation X such that $XR \subseteq S$. So, we have the Galois connection $XR \subseteq S$ if and only if $X \subseteq S/R$, for all relations X. Similarly, the *right residual* of S over R, in symbols $R \backslash S$, is the greatest relation X such that $RX \subseteq S$. This implies that $RX \subseteq S$ if and only if $X \subseteq R \backslash S$, for all relations X. We will also need relations which are left and right residuals simultaneously. The *symmetric quotient* $R \div S$ of two relations R and S is defined as the greatest relation X such that $RX \subseteq S$ and $XS^{\mathsf{c}} \subseteq R^{\mathsf{c}}$. In terms of the basic operations we have $S/R = \overline{\overline{S}R^{\mathsf{c}}}$, $R \setminus S = \overline{R^{\mathsf{c}}\overline{S}}$ and $R \div S = (R \setminus S) \cap (R^{\mathsf{c}}/S^{\mathsf{c}})$, for all relations R and S.

Besides empty relations, universal relations and identity relations, we need further basic relations which specify fundamental set-theoretic constructions. Assume A to be a set and let 2^A denote its powerset. Then the *membership relation* $\mathsf{E} : A \leftrightarrow 2^A$ is the relation-level equivalent to the set-theoretic predicate '$\in$'. Hence, we have $\mathsf{E}_{x,Y}$ if and only if $x \in Y$, for all $x \in A$ and $Y \in 2^A$. With the help of E we can introduce two relations on 2^A via $\mathsf{S} := \mathsf{E} \setminus \mathsf{E} : 2^A \leftrightarrow 2^A$ and $\mathsf{C} := \mathsf{E} \div \overline{\mathsf{E}} : 2^A \leftrightarrow 2^A$. A little component-wise calculation shows $\mathsf{S}_{X,Y}$ if and only if $X \subseteq Y$ and $\mathsf{C}_{X,Y}$ if and only if $Y = \overline{X}$, for all $X \in 2^A$ and $Y \in 2^A$, where $\overline{X}$ is the complement of the set X relative to its superset A. Therefore, we call S a *subset relation* and C a *set complement relation*.

3 Fundamentals of Multirelations

In this section we recall basic definitions, operations and properties of multirelations and express them in terms of relations. The presentation follows [15].

A *multirelation* (as introduced in [19,23]) is a relation of type $A \leftrightarrow 2^B$ in the sense of Section 2. It maps an element of A to a set of subsets of B. Union, intersection and complement apply to multirelations as to relations. Particular multirelations are empty relations $\mathsf{O} : A \leftrightarrow 2^B$, universal relations $\mathsf{T} : A \leftrightarrow 2^B$ and membership relations $\mathsf{E} : A \leftrightarrow 2^A$. The composition of the multirelations $Q : A \leftrightarrow 2^B$ and $R : B \leftrightarrow 2^C$ is the multirelation $Q\,;R : A \leftrightarrow 2^C$, given by

$$(Q\,;R)_{x,Z} \iff \exists Y \in 2^B : Q_{x,Y} \wedge \forall y \in Y : R_{y,Z},$$

for all $x \in A$ and $Z \in 2^C$. The *dual* of a multirelation $R : A \leftrightarrow 2^B$ is the multirelation $R^{\mathsf{d}} : A \leftrightarrow 2^B$ given by

$$R^{\mathsf{d}}{}_{x,Y} \iff \neg R_{x,\overline{Y}},$$

for all $x \in A$ and $Y \in 2^B$, where $\overline{Y}$ is the complement of Y relative to its superset B. Dual has higher precedence than composition, which has higher precedence than union and intersection. A multirelation $R : A \leftrightarrow 2^B$ is *up-closed* if

$$R_{x,Y} \wedge Y \subseteq Z \implies R_{x,Z}$$

for all $x \in A$ and $Y, Z \in 2^B$. This means that if an element of A is related to a set Y, it also has to be related to all supersets of Y. By $A \overset{\mathsf{u}}{\leftrightarrow} 2^B$ we denote the set of all up-closed multirelations of type $A \leftrightarrow 2^B$.

The following result expresses multirelational composition, the dual and the property of being up-closed in terms of relation-algebraic operations and constants, namely right residual, membership relations, set complement relations C and subset relations S. It is proved in [15, Theorems 2, 4 and 6]; see also [16,25].

Theorem 1. *Let $Q : A \leftrightarrow 2^B$ and $R : B \leftrightarrow 2^C$ be multirelations. Then we have $Q ; R = Q(\mathsf{E} \setminus R)$ and $Q^{\mathsf{d}} = \overline{Q}\mathsf{C} = \overline{Q\mathsf{C}}$. Furthermore, Q is up-closed if and only if $Q = Q\mathsf{S}$.*

A multirelation $R : A \leftrightarrow 2^A$ models a two-player game as shown in [19]. The set A describes the possible states of the game. For each state $x \in A$ the set of subsets $Ys = \{Y \in 2^A \mid R_{x,Y}\}$ to which x is related gives the options of the first player. The first player chooses one of these subsets, a set $Y \in Ys$. This set Y gives the options of the second player, who chooses one of its elements $y \in Y$, which is the next state of the game. If the first player cannot make a choice because Ys is empty, the second player wins. If the second player cannot make a choice because Y is empty, the first player wins. Multirelations can also be used to describe the interaction of two agents in a computation (see [5,10,17]), certain kinds of contact (see [1,4]) and concurrency (see [21]).

Being relations, the multirelations of type $A \leftrightarrow 2^B$ form a bounded distributive lattice under the operations of union and intersection. The structure becomes more diversified once we take composition into account. First, familiar laws of relation algebras – that composition distributes over union and has the empty relation as a zero – no longer hold from both sides, but just from one side. Second, other laws of relation algebras – that composition is associative and has the identity relation as a neutral element – hold for up-closed multirelations, but need to be weakened in the general case as shown in [11]. On the other hand, composition remains $\subseteq$-isotone. These and related properties are summarised in the following result.

Theorem 2. *For all multirelations P, Q and R we have*

(1) $\mathsf{O} ; R = \mathsf{O}$ *(2)* $\mathsf{E} ; R = R$ *(3)* $\mathsf{T} ; R = \mathsf{T}$ *(4)* $R \subseteq R ; \mathsf{E}$,

where in (4) equality holds if and only if R is up-closed, and also

(5) $(P \cup Q) ; R = P ; R \cup Q ; R$, *(6)* $(P \cap Q) ; R \subseteq P ; R \cap Q ; R$,

where in (6) equality holds if P and Q are up-closed, and also

(7) $(P ; Q) ; R \subseteq P ; (Q ; R)$,

where in (7) equality holds if Q is up-closed, and finally

(8) $P ; Q \cup P ; R \subseteq P ; (Q \cup R)$ *(9)* $P ; (Q \cap R) \subseteq P ; Q \cap P ; R$.

Proof. All properties are proved in [15, Theorems 3 and 7] except (4) and (7) for general multirelations. A proof of (4) is $R \subseteq R\mathsf{S} = R(\mathsf{E} \setminus \mathsf{E}) = R ; \mathsf{E}$. To prove (7) we use that $\mathsf{E}(\mathsf{E} \setminus Q)(\mathsf{E} \setminus R) \subseteq Q(\mathsf{E} \setminus R)$ implies $(\mathsf{E} \setminus Q)(\mathsf{E} \setminus R) \subseteq \mathsf{E} \setminus (Q(\mathsf{E} \setminus R))$ by the Galois connection. Hence, we get the result as follows:

$$\begin{aligned}(P ; Q) ; R &= (P ; Q)(\mathsf{E} \setminus R) = P(\mathsf{E} \setminus Q)(\mathsf{E} \setminus R) \\ &\subseteq P(\mathsf{E} \setminus (Q(\mathsf{E} \setminus R))) = P(\mathsf{E} \setminus (Q ; R)) = P ; (Q ; R)\end{aligned}$$ $\square$

The dual operation reverses the lattice order and distributes over composition of up-closed multirelations. Again this needs to be weakened in the general case. These and further properties are summarised in the following result.

Theorem 3. *For all multirelations Q and R we have*

$$\text{(1) } \mathsf{O}^{\mathsf{d}} = \mathsf{T} \qquad \text{(2) } \mathsf{E}^{\mathsf{d}} = \mathsf{E} \qquad \text{(3) } \mathsf{T}^{\mathsf{d}} = \mathsf{O} \qquad \text{(4) } R^{\mathsf{dd}} = R,$$

and also

$$\begin{array}{ll} \text{(5) } (Q \cup R)^{\mathsf{d}} = Q^{\mathsf{d}} \cap R^{\mathsf{d}} & \text{(7) } (Q \,;R)^{\mathsf{d}} \subseteq Q^{\mathsf{d}} \,; R^{\mathsf{d}} \\ \text{(6) } (Q \cap R)^{\mathsf{d}} = Q^{\mathsf{d}} \cup R^{\mathsf{d}} & \text{(8) } (Q \,;R)^{\mathsf{d}} = (Q \,;\mathsf{E})^{\mathsf{d}} \,; R^{\mathsf{d}}, \end{array}$$

where in (7) equality holds if Q is up-closed.

Proof. All properties are proved in [15, Theorems 5 and 7] except (7) and (8) for general multirelations. A proof of (7) and (8) is as follows:

$$\begin{aligned} (Q\,;R)^{\mathsf{d}} &= \overline{Q\,;\overline{R}}\mathsf{C} = \overline{Q(\mathsf{E}\setminus \overline{R})}\mathsf{C} = \overline{Q\mathsf{S}(\mathsf{E}\div \overline{R})}\mathsf{C} = \overline{Q\mathsf{S}}(\mathsf{E}\div R)\mathsf{C} = \overline{Q\mathsf{S}}(\overline{\mathsf{E}}\div \overline{R})\mathsf{C} \\ &= \overline{Q\mathsf{S}}(\overline{\mathsf{E}}\div \mathsf{E})(\mathsf{E}\div \overline{R})\mathsf{C} = \overline{Q\mathsf{S}}\mathsf{C}^{\mathsf{c}}(\mathsf{E}\div \overline{R})\mathsf{C} = \overline{Q\mathsf{S}}\mathsf{C}(\mathsf{E}\div \overline{R})\mathsf{C} \\ &= \overline{Q\mathsf{S}}\mathsf{S}^{\mathsf{c}}\mathsf{C}(\mathsf{E}\div \overline{R})\mathsf{C} = \overline{Q\mathsf{S}}\mathsf{C}\mathsf{S}(\mathsf{E}\div \overline{R})\mathsf{C} = \overline{Q\mathsf{S}}\mathsf{C}(\mathsf{E}\setminus \overline{R})\mathsf{C} = (Q\mathsf{S})^{\mathsf{d}}(\mathsf{E}\setminus \overline{R})\mathsf{C} \\ &= (Q\mathsf{S})^{\mathsf{d}}(\mathsf{E}\setminus (\overline{R}\mathsf{C})) = (Q(\mathsf{E}\setminus \mathsf{E}))^{\mathsf{d}}(\mathsf{E}\setminus R^{\mathsf{d}}) = (Q\,;\mathsf{E})^{\mathsf{d}}\,;R^{\mathsf{d}} \subseteq Q^{\mathsf{d}}\,;R^{\mathsf{d}} \end{aligned}$$

This calculation uses $\overline{Q\mathsf{S}}\mathsf{S}^{\mathsf{c}} = \overline{Q\mathsf{S}}$. The inclusion '$\subseteq$' follows by applying a Schröder equivalence to $Q\mathsf{SS} \subseteq Q\mathsf{S}$ and the inclusion '$\supseteq$' follows from $\mathsf{I} \subseteq \mathsf{S}$. See the proof of [15, Theorem 7.3] for an explanation of the other steps. □

4 Algebraic Structures for Investigating Multirelations

In this section we capture the properties of multirelations shown in Section 2 by five algebraic structures, which are introduced in the following.

A *bounded join-semilattice* is an algebraic structure $(S, +, 0)$ satisfying for all $x, y, z \in S$ the associativity, commutativity, idempotence and neutrality axioms:

$$x + (y + z) = (x + y) + z \qquad x + y = y + x \qquad x + x = x \qquad 0 + x = x$$

The *semilattice order*, defined by $x \leq y$ if and only if $x + y = y$, for all $x, y \in S$, has the least element 0 and the least upper bound operation '+'. The operation '+' is $\leq$-isotone.

Next, a *bounded distributive lattice* $(S, +, \curlywedge, 0, \top)$ adds to a bounded join-semilattice a dual bounded meet-semilattice $(S, \curlywedge, \top)$ as well as distribution and absorption axioms, such that for all $x, y, z \in S$ the following equations hold:

$$\begin{array}{rl} x \curlywedge (y \curlywedge z) = (x \curlywedge y) \curlywedge z & x + (y \curlywedge z) = (x + y) \curlywedge (x + z) \\ x \curlywedge y = y \curlywedge x & x \curlywedge (y + z) = (x \curlywedge y) + (x \curlywedge z) \\ x \curlywedge x = x & x + (x \curlywedge y) = x \\ \top \curlywedge x = x & x \curlywedge (x + y) = x \end{array}$$

The semilattice order has the alternative characterisation that $x \leq y$ if and only if $x \curlywedge y = x$, for all $x, y \in S$, the greatest element $\top$ and the greatest lower bound operation '$\curlywedge$'. The operation '$\curlywedge$' is $\leq$-isotone.

A *pre-left semiring* $(S, +, \cdot, 0, 1)$ expands a bounded join-semilattice $(S, +, 0)$ with a binary operation '$\cdot$' and a constant 1 with the following axioms for all $x, y, z \in S$:

$$\begin{array}{cc} x = 1 \cdot x & (x \cdot y) + (x \cdot z) \leq x \cdot (y + z) \\ x \leq x \cdot 1 & (x \cdot z) + (y \cdot z) = (x + y) \cdot z \\ (x \cdot y) \cdot z \leq x \cdot (y \cdot z) & 0 = 0 \cdot x \end{array}$$

Note the inequalities in the left column. The operation '$\cdot$' is $\leq$-isotone. We often abbreviate a product $x \cdot y$ via juxtaposition to xy.

An *idempotent left semiring* (see [18]) is a pre-left semiring $(S, +, \cdot, 0, 1)$ whose reduct $(S, \cdot, 1)$ is a monoid, which is enforced by adding the axioms

$$x = x \cdot 1 \qquad (x \cdot y) \cdot z = x \cdot (y \cdot z),$$

for all $x, y, z \in S$. Idempotent semirings are rings in which the operation '$+$' is idempotent instead of having an inverse. Idempotent left semirings are idempotent semirings in which the operation '$\cdot$' is $\leq$-isotone instead of distributing over the operation '$+$' from the left and having the right zero 0. Pre-left semirings further weaken idempotent left semirings by requiring only one inequality of the associativity and right-neutral properties. This is because multirelations do not satisfy the other inequalities in general.

Finally, combining the lattice and semiring operations, an *M0-algebra* is an algebraic structure $(S, +, \cdot, \curlywedge, 0, 1, \top)$ such that the reduct $(S, +, \curlywedge, 0, \top)$ is a bounded distributive lattice and the reduct $(S, +, \cdot, 0, 1)$ is a pre-left semiring.

The algebraic results we will derive in the following sections apply to multirelations because of the following instances. The multirelations over a set A form a bounded distributive lattice $(A \leftrightarrow 2^A, \cup, \cap, \mathsf{O}, \mathsf{T})$. By Theorem 2 these multirelations also form an M0-algebra $(A \leftrightarrow 2^A, \cup, ;, \cap, \mathsf{O}, \mathsf{E}, \mathsf{T})$ and the subset of up-closed multirelations forms an idempotent left semiring $(A \overset{\mathrm{u}}{\leftrightarrow} 2^A, \cup, ;, \mathsf{O}, \mathsf{E})$. We refer to [22,29] for further algebraic structures underlying up-closed multirelations and to [16] for placing them in a categorical setting. See also [21], where another kind of multirelational composition '$\cdot$' is introduced that gives rise to an M0-algebra. As shown in [12], this operation is not associative for general multirelations, but satisfies $(P \cdot Q) \cdot R \subseteq P \cdot (Q \cdot R)$ and $P = P \cdot 1$ for all P, Q and R, where $1 = \mathsf{I} \div \mathsf{E}$ is the singleton multirelation.

5 Reflexive-Transitive Closures of Multirelations

As proved in [11], multirelational composition has a left residual. If we define it by $R /\!\!/ Q := R / (\mathsf{E} \setminus Q)$, for all multirelations R and Q, then we get

$$P ; Q \subseteq R \iff P(\mathsf{E} \setminus Q) \subseteq R \iff P \subseteq R / (\mathsf{E} \setminus Q) \iff P \subseteq R /\!\!/ Q,$$

for all multirelations P, Q and R. In this section we use left residuals and an appropriate algebraic structure to relate three different representations of reflexive-transitive closures of multirelations.

A *residuated pre-left semiring* $(S, +, \cdot, /, 0, 1)$ expands a pre-left semiring $(S, +, \cdot, 0, 1)$ with a binary operation '/' satisfying the Galois connection

$$xy \le z \iff x \le z/y,$$

for all $x, y, z \in S$. It follows that the operation '/' is $\le$-isotone in its first argument and $\le$-antitone in its second argument. Moreover, we obtain the two properties $(x/y)y \le x$ and $x/1 \le x$, for all $x, y \in S$. As a consequence we get the following instance. The multirelations over a set A form a residuated pre-left semiring $(A \leftrightarrow 2^A, \cup, ;, /\!\!/, \mathsf{O}, \mathsf{E})$.

The $\le$-isotone functions f, g and h of the following result capture left recursion, right recursion and symmetric recursion, respectively. The $\le$-least prefixpoint μf of the function f is axiomatised using its unfold and induction properties, that is, $f(\mu f) \le \mu f$ and that $f(x) \le x$ implies $\mu f \le x$, for all $x \in S$. Similar axioms are assumed for μg and μh. It is known that left and right recursion coincide for relations, but in general they do not for multirelations.

Theorem 4. *Let S be a residuated pre-left semiring and let $y \in S$. Depending on y, let f, g and h be functions on S defined by*

$$f(x) = 1 + x \cdot y \qquad g(x) = 1 + y \cdot x \qquad h(x) = 1 + y + x \cdot x,$$

for all $x \in S$. Assume that μf, μg and μh exist. Then we have $\mu f \le \mu g = \mu h$.

Proof. We first show $\mu f \le \mu g$. Semi-associativity of composition, the Galois property of the left residual and the prefixpoint property of μg imply

$$(y \cdot (\mu g/y)) \cdot y \le y \cdot ((\mu g/y) \cdot y) \le y \cdot \mu g \le 1 + y \cdot \mu g \le \mu g.$$

Hence, we get $y \cdot (\mu g/y) \le \mu g/y$. Moreover, $1 \le 1 + y \cdot \mu g \le \mu g$ holds, whence semi-neutrality of composition gives

$$1 \cdot y = y \le y \cdot 1 \le 1 + y \cdot \mu g \le \mu g.$$

So, $1 \le \mu g/y$ and, together, we have

$$g(\mu g/y) = 1 + y \cdot (\mu g/y) \le \mu g/y.$$

From this we obtain $\mu g \le \mu g/y$ by the least prefixpoint property of μg. Hence

$$f(\mu g) = 1 + \mu g \cdot y \le \mu g$$

and, therefore, $\mu f \le \mu g$ follows by the least prefixpoint property of μf.

We next show $\mu g \le \mu h$. This part does not use residuals. From the least prefixpoint property of μh we get $y \le 1 + y + \mu h \cdot \mu h = h(\mu h) \le \mu h$; hence

$$g(\mu h) = 1 + y \cdot \mu h \le 1 + y + \mu h \cdot \mu h = h(\mu h) \le \mu h$$

by the prefixpoint property of μh. Therefore, we arrive at $\mu g \le \mu h$ by the least prefixpoint property of μg.

We finally show $\mu h \le \mu g$ following the argument of [6, Satz 10.1.5], which is for homogeneous relations. Semi-associativity of composition, a property of the left residual and the unfold property of μg imply:

$$\begin{aligned} g(\mu g/\mu g) \cdot \mu g &= (1 + y \cdot (\mu g/\mu g)) \cdot \mu g = 1 \cdot \mu g + (y \cdot (\mu g/\mu g)) \cdot \mu g \\ &\le \mu g + y \cdot ((\mu g/\mu g) \cdot \mu g) \le \mu g + 1 + y \cdot \mu g = \mu g + g(\mu g) = \mu g \end{aligned}$$

As a consequence we obtain $g(\mu g/\mu g) \le \mu g/\mu g$ and this leads to $\mu g \le \mu g/\mu g$ by the least prefixpoint property of μg, whence $\mu g \cdot \mu g \le \mu g$. With $1 \le \mu g$ and $y \le \mu g$ shown above, it follows that

$$h(\mu g) = 1 + y + \mu g \cdot \mu g \le \mu g.$$

Therefore we have $\mu h \le \mu g$ by the least prefixpoint property of μh. ◻

For up-closed multirelations the equality $\mu g = \mu h$ is shown in [28]. Furthermore, for finitary up-closed multirelations $\bigcup_{n \in \mathbb{N}} g^n(\mathsf{O}) \subseteq \mu h$ is shown in [13] and $\bigcup_{n \in \mathbb{N}} g^n(\mathsf{O}) = \mu g$ is shown in [11].

We proved Theorem 4 also in Isabelle/HOL using its integrated automated theorem provers and SMT solvers, which are described in [8,20]. The same holds for the theorems we will present in the next two sections, that is, Theorem 5 to Theorem 8. We therefore omit their proofs, which are given in the Isabelle theory files available at http://www.csse.canterbury.ac.nz/walter.guttmann/algebra/.

6 Properties of Multirelations

A number of properties of multirelations were used in previous work for modelling games, protocols, computations, contact, closure and topology, see [1,5,17,19,23], for example. Algebraic definitions of these and other properties are summarised in Figure 1. Its second column states the property in terms of relations and the third column gives the corresponding definition in M0-algebras. The distributivity properties universally quantify over the multirelations P, Q and the elements y, z of the M0-algebra, respectively.

For up-closed multirelations several of the properties listed in Figure 1 are dual to each other, that is, can be obtained by applying the multirelational dual operation. This does not hold for general multirelations: for example, the conjunction of reflexive and transitive implies up-closed, but the conjunction of their duals co-reflexive and dense does not imply up-closed, which is self-dual.

In this section we investigate the connections between the properties in Figure 1 using the algebraic structure of multirelations. While many results can be derived in M0-algebras, additional axioms are needed to prove some others, leading to the following new algebraic structure. An *M1-algebra* is an M0-algebra $(S, +, \cdot, \curlywedge, 0, 1, \top)$ satisfying the axioms

$$\top = \top x \qquad x(yz) = (x(y1))z \qquad xz \curlywedge yz = (x1 \curlywedge y1)z,$$

R or x is ...	if and only if	algebraically
total	$R;\mathsf{T} = \mathsf{T}$	$x\top = \top$
co-total	$R;\mathsf{O} = \mathsf{O}$	$x0 = 0$
transitive	$R;R \subseteq R$	$xx \leq x$
dense	$R \subseteq R;R$	$x \leq xx$
reflexive	$\mathsf{E} \subseteq R$	$1 \leq x$
co-reflexive	$R \subseteq \mathsf{E}$	$x \leq 1$
idempotent	$R;R = R$	$xx = x$
up-closed	$R;\mathsf{E} = R$	$x1 = x$
$\cup$-distributive	$R;(P \cup Q) = R;P \cup R;Q$	$x(y + z) = xy + xz$
$\cap$-distributive	$R;(P \cap Q) = R;P \cap R;Q$	$x(y \curlywedge z) = xy \curlywedge xz$
a contact	$R;R \cup \mathsf{E} = R$	$xx + 1 = x$
a kernel	$R;R \cap \mathsf{E} = R;\mathsf{E}$	$xx \curlywedge 1 = x1$
a test	$R;\mathsf{T} \cap \mathsf{E} = R$	$x\top \curlywedge 1 = x$
a co-test	$R;\mathsf{O} \cup \mathsf{E} = R$	$x0 + 1 = x$
a vector	$R;\mathsf{T} = R$	$x\top = x$

Fig. 1. Fundamental properties

for all $x, y, z \in S$. An equivalent structure is obtained if just '$\leq$' is assumed instead of equality in each axiom. If all elements are up-closed, that is, $x1 = x$ holds for all $x \in S$, the last two axioms collapse to associativity of the operation '$\cdot$' and right-distributivity of '$\cdot$' over the operation '$\curlywedge$'. This shows how to obtain weaker axioms which hold for all multirelations. The following theorem summarises our results about relationships between the properties in Figure 1.

Theorem 5. *The implications shown in Figure 2 drawn as continuous (dashed) arrows hold in M0-algebras (M1-algebras). Furthermore, arrows originating in the same point indicate that the property is equivalent to the conjunction of the targets.*

Moreover, in all M1-algebras S the vector property $x\top = \top$ is equivalent to its dual $x0 = 0$ for all $x \in S$.

7 Closure Properties of Multirelational Operations

It is known that up-closed multirelations are closed under the multirelational operations we have introduced in Section 3. In this section we systematically investigate the closure properties for certain classes of multirelations, which are given by the properties presented in Figure 1. For dealing with the dual operation we need additional axioms, which lead to the expansions of M0-algebras we will introduce in this section.

First, an *M2-algebra* $(S, +, \cdot, \curlywedge, ^{\mathsf{d}}, 0, 1, \top)$ is an M0-algebra $(S, +, \cdot, \curlywedge, 0, 1, \top)$ expanded with a unary dual operation '$^{\mathsf{d}}$' satisfying the axioms

$$(xy)^{\mathsf{d}} = (x1)^{\mathsf{d}} y^{\mathsf{d}} \qquad (x + y)^{\mathsf{d}} = x^{\mathsf{d}} \curlywedge y^{\mathsf{d}} \qquad x^{\mathsf{dd}} = x \qquad 1^{\mathsf{d}} = 1,$$

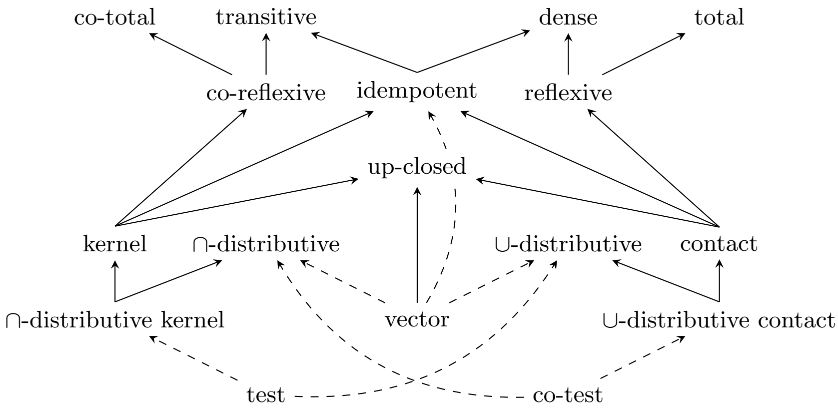

Fig. 2. Relationships between the fundamental properties

for all $x, y \in S$. Note again how distributivity of the operation '$^\mathsf{d}$' over the operation '$\cdot$', which holds for up-closed multirelations, is weakened by replacing x with $x1$. The above axioms imply the additional axioms of M1-algebras. Thus, we obtain the following result.

Theorem 6. *All M2-algebras are M1-algebras.*

For reasoning about up-closed multirelations we use that the operation '$^\mathsf{d}$' distributes over the operation '$\cdot$'. As a further expansion of M0-algebras, therefore, an *M3-algebra* $(S, +, \cdot, \curlywedge, ^\mathsf{d}, 0, 1, \top)$ is an M0-algebra $(S, +, \cdot, \curlywedge, 0, 1, \top)$ expanded with a unary dual operation '$^\mathsf{d}$' satisfying the axioms

$$(xy)^\mathsf{d} = x^\mathsf{d} y^\mathsf{d} \qquad (x+y)^\mathsf{d} = x^\mathsf{d} \curlywedge y^\mathsf{d} \qquad x^{\mathsf{dd}} = x \qquad 1^\mathsf{d} = 1,$$

for all $x, y \in S$. These axioms imply the axioms of M2-algebras. Moreover, we obtain that the operation '$\cdot$' is associative with right-neutral element 1, that is, the idempotent left semiring structure.

Theorem 7. *All M3-algebras are M2-algebras and idempotent left semirings.*

The algebraic results obtained so far apply to multirelations due to the following instances. By Theorem 3, the multirelations over a set A form an M2-algebra $(A \leftrightarrow 2^A, \cup, ;, \cap, ^\mathsf{d}, \mathsf{O}, \mathsf{E}, \mathsf{T})$ and the up-closed multirelations over A form an M3-algebra $(A \overset{u}{\leftrightarrow} 2^A, \cup, ;, \cap, ^\mathsf{d}, \mathsf{O}, \mathsf{E}, \mathsf{T})$. The next theorem summarises the closure properties of multirelations.

Theorem 8. *Figure 3 shows which properties in Figure 1 hold for the multirelational constants and with respect to which operations these properties are closed. There an entry ■ (□) means that the property is closed under the respective operation in M2-algebras (M3-algebras). All ■ entries except those for the operation*

	O	E	T	∪	∩	;	d
total	–	■	■	■	□	□	▽
co-total	■	■	–	■	■	■	▲
transitive	■	■	■	–	■	–	▼
dense	■	■	■	■	–	–	△
reflexive	–	■	■	■	■	■	▼
co-reflexive	■	■	–	■	■	■	▲
idempotent	■	■	■	–	–	–	□
up-closed	■	■	■	■	■	■	■
∪-distributive	■	■	■	■	–	□	▽
∩-distributive	■	■	■	–	□	□	△
a contact	–	■	■	–	■	–	▼
a kernel	■	■	–	■	–	–	▲
a ∪-distributive contact	–	■	■	–	–	–	▼
a ∩-distributive kernel	■	■	–	–	–	–	▲
a test	■	■	–	■	■	■	▼
a co-test	–	■	■	■	■	■	▲
a vector	■	–	■	■	■	■	■

Fig. 3. Closure properties of multirelations

'd' *follow in M1-algebras; most of these follow already in M0-algebras. An entry* ▼/▲ (▽/△) *means that if x satisfies the property then x^{d} satisfies the property below/above in M2-algebras (M3-algebras). An entry – means that the property is not closed under the respective operation even for up-closed multirelations.*

To give an example, the dual of a co-total multirelation is total and the dual of an up-closed total multirelation is co-total. Another consequence of the closure properties are sub-algebras. For example, the set of co-total multirelations forms a pre-left semiring and so does the set of co-reflexive multirelations.

It is unknown if any of the findings □ can be strengthened to ■ in the rows for ∪-/∩-distributive in Figure 3. Moreover, it is unknown if the finding △ can be strengthened to ▲ in the row for ∩-distributive. Counterexamples for the other claims are shown in Figures 4, 5 and 6 as Boolean matrices (where a grey square denotes a 1-entry and a white square denotes a 0-entry). Most counterexamples have been found using a Haskell program which performs an exhaustive search. For ∪- and ∩-distributivity of up-closed multirelations we use the alternative characterisation provided by Aumann contacts given in Section 8.

Note that M2-algebras are not complete for multirelations. The counterexample generator Nitpick, which is described in [9], finds a counterexample showing that $x\top \curlywedge yz \leq (x\top \curlywedge y)z$ does not follow in M2-algebras. However, this property holds for multirelations since

$$P\,;\mathsf{T} \cap Q\,;R = P\mathsf{T} \cap Q(\mathsf{E} \setminus R) = (P\mathsf{T} \cap Q)(\mathsf{E} \setminus R) = (P\,;\mathsf{T} \cap Q)\,;R.$$

This calculation uses that $P\,;\mathsf{T} = P\mathsf{T}$ as shown in [15], so intersection with this vector can be imported into the first argument of a composition.

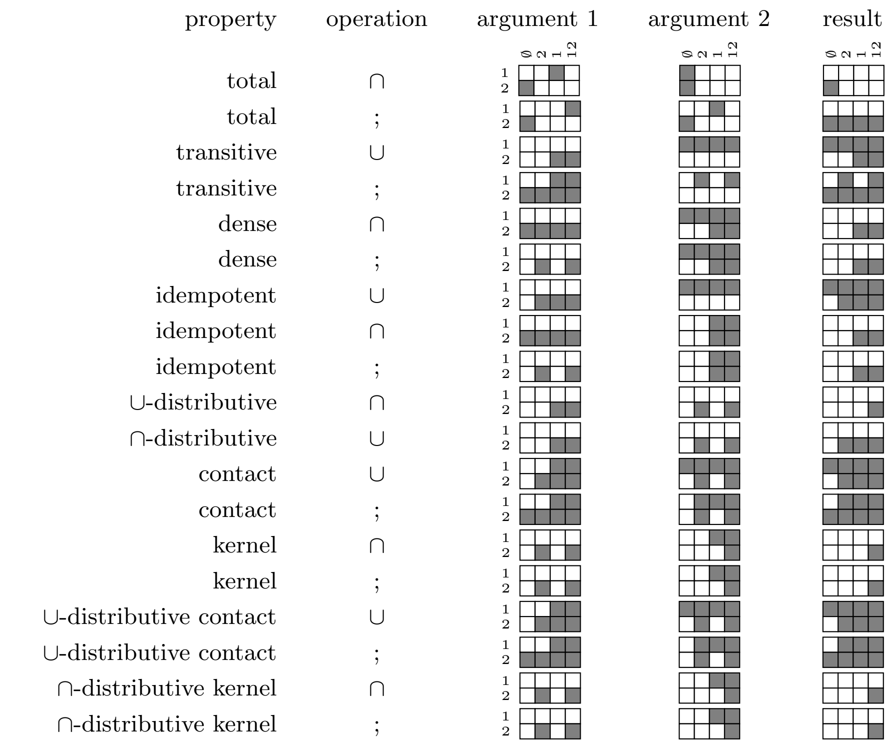

Fig. 4. Counterexamples generated by a Haskell program

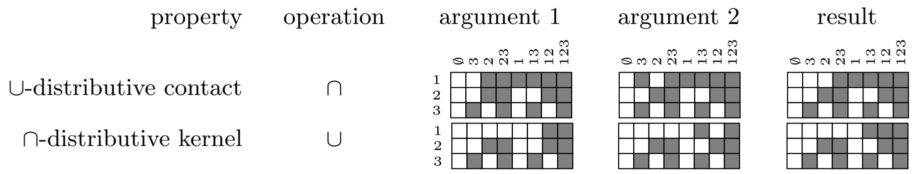

Fig. 5. Manually generated counterexamples

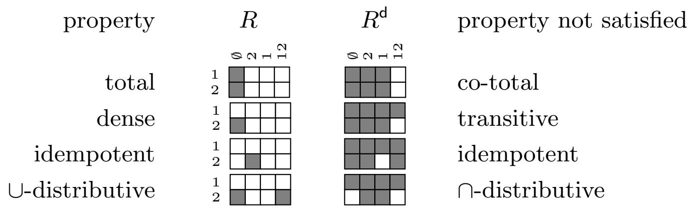

Fig. 6. Counterexamples for the operation ‘d’ generated by a Haskell program

Neither are M3-algebras complete for up-closed multirelations. Nitpick shows that $x\top \curlywedge x^{\mathsf{d}}0 = 0$ does not follow in M3-algebras, although it holds for up-closed multirelations. To see this, note that it is an axiom of 'algebras of monotonic Boolean transformers' of [22] or consider the following proof. Let R be an up-closed multirelation. Then we have $R(\mathsf{E} \setminus \mathsf{E}) = R;\mathsf{E} = R$. By a Schröder equivalence we get $R^{\mathsf{c}}\overline{R} \subseteq \mathsf{E}^{\mathsf{c}}\overline{\mathsf{E}} \subseteq \mathsf{T}\overline{\mathsf{E}}$. Hence, $\mathsf{T}R^{\mathsf{c}}\overline{R}\mathsf{C} \subseteq \mathsf{T}\overline{\mathsf{E}}\mathsf{C} = \mathsf{TE}$. Another Schröder equivalence gives $\overline{R}\mathsf{C}\overline{\mathsf{E}^{\mathsf{c}}\mathsf{T}} \subseteq \overline{R\mathsf{T}}$. So, the desired result is shown by

$$R;\mathsf{T} \cap R^{\mathsf{d}};\mathsf{O} = R\mathsf{T} \cap R^{\mathsf{d}}(\mathsf{E} \setminus \mathsf{O}) = R\mathsf{T} \cap \overline{R}\mathsf{C}\overline{\mathsf{E}^{\mathsf{c}}\mathsf{T}} \subseteq R\mathsf{T} \cap \overline{R\mathsf{T}} = \mathsf{O}.$$

8 Aumann Contacts and Multirelational Properties

In [1,2,3,4] G. Aumann investigated certain laws for modelling the notion of a contact in topology. Translated into the language of multirelations, he considered for a multirelation $R : A \leftrightarrow 2^A$ the following five axioms:

$$\begin{array}{ll}
(K_0) & \neg\exists x \in A : R_{x,\emptyset} \\
(K_1) & \forall x \in A : R_{x,\{x\}} \\
(K_2) & \forall x \in A : \forall Y, Z \in 2^A : R_{x,Y} \wedge Y \subseteq Z \Rightarrow R_{x,Z} \\
(K_3) & \forall x \in A : \forall Y, Z \in 2^A : R_{x,Y} \wedge (\forall y \in Y : R_{y,Z}) \Rightarrow R_{x,Z} \\
(K_4) & \forall x \in A : \forall Y, Z \in 2^A : R_{x,Y\cup Z} \Leftrightarrow R_{x,Y} \vee R_{x,Z}
\end{array}$$

Aumann called multirelations satisfying the formulas (K_1) to (K_3) 'contact relations' and multirelations satisfying the formulas (K_0) to (K_4) 'topological contact relations'. In this section we give multirelation-algebraic characterisations of these logical formulas. See [26] for the relation-algebraic treatment of a correspondence between contact relations and closure operations. Axioms (K_0), (K_2) and (K_4) generalise to multirelations of type $A \leftrightarrow 2^B$ in a straight-forward way. The following result gives the property corresponding to K_0.

Theorem 9. *A multirelation satisfies* (K_0) *if and only if it is co-total.*

Proof. Axiom (K_0) applied to a multirelation $R : A \leftrightarrow 2^B$ elaborates as follows:

$$\begin{aligned}
\neg\exists x \in A : R_{x,\emptyset} &\iff \forall x \in A : \neg R_{x,\emptyset} \\
&\iff \forall x \in A : \forall X \in 2^B : R_{x,X} \Rightarrow X \neq \emptyset \\
&\iff \forall x \in A : \forall X \in 2^B : R_{x,X} \Rightarrow \exists y \in B : y \in X \\
&\iff \forall x \in A : \forall X \in 2^B : R_{x,X} \Rightarrow \exists y \in B : \mathsf{T}_{x,y} \wedge \mathsf{E}_{y,X} \\
&\iff \forall x \in A : \forall X \in 2^B : R_{x,X} \Rightarrow (\mathsf{TE})_{x,X} \\
&\iff R \subseteq \mathsf{TE} \\
&\iff \mathsf{T}R \subseteq \mathsf{TE} \\
&\iff R\overline{\mathsf{TE}}^{\mathsf{c}} \subseteq \mathsf{O} \\
&\iff R(\mathsf{E} \setminus \mathsf{O}) \subseteq \mathsf{O} \\
&\iff R;\mathsf{O} \subseteq \mathsf{O}
\end{aligned}$$

Hence, the characterisation in Figure 1 shows the claim. □

The forward implication of this theorem is stated in [24], where such multirelations are called 'total'. We call the above property 'co-total' to keep the standard use of 'total' known from relations and functions. Namely,

$$R\,;\mathsf{T} = R(\mathsf{E} \setminus \mathsf{T}) = R\overline{\mathsf{E}^{\mathsf{c}}\overline{\overline{\mathsf{T}}}} = R\overline{\mathsf{E}^{\mathsf{c}}\mathsf{O}} = R\overline{\mathsf{O}} = R\mathsf{T}$$

implies that the multirelation-algebraic property $R\,;\mathsf{T} = \mathsf{T}$ is equivalent to the relation-algebraic property of totality $R\mathsf{T} = \mathsf{T}$. In [23] multirelations R satisfying the property $R\,;\mathsf{T} = \mathsf{T}$ are called 'proper'. Next, we investigate axiom (K_1) and relate it to a property in Figure 1.

Theorem 10. *Every reflexive multirelation satisfies* (K_1)*. An up-closed multirelation satisfies* (K_1) *if and only if it is reflexive.*

Proof. Axiom (K_1) applied to a multirelation $R : A \leftrightarrow 2^A$ elaborates as follows:

$$\begin{aligned}
\forall x \in A : R_{x,\{x\}} &\iff \forall x \in A : \forall X \in 2^A : \{x\} = X \Rightarrow R_{x,X} \\
&\Longleftarrow \forall x \in A : \forall X \in 2^A : \{x\} \subseteq X \Rightarrow R_{x,X} \\
&\iff \forall x \in A : \forall X \in 2^A : x \in X \Rightarrow R_{x,X} \\
&\iff \forall x \in A : \forall X \in 2^A : \mathsf{E}_{x,X} \Rightarrow R_{x,X} \\
&\iff \mathsf{E} \subseteq R
\end{aligned}$$

Again Figure 1 shows the first claim. If R is up-closed, then the reverse implication holds since $R_{x,\{x\}}$ and $\{x\} \subseteq X$ imply $R_{x,X}$. □

Axiom (K_2) is the logical characterisation of R being an up-closed multirelation. The relation-algebraic characterisation $R = R\mathsf{S}$ is shown in [15, Theorem 6] and the multirelation-algebraic characterisation $R;\mathsf{E} = R$ in [15, Theorem 7.1]. With respect to axiom (K_3), we have the following correspondence.

Theorem 11. *A multirelation satisfies* (K_3) *if and only if it is transitive.*

Proof. Axiom (K_3) applied to a multirelation $R : A \leftrightarrow 2^A$ elaborates as follows:

$$\begin{aligned}
&\forall x \in A : \forall Y, Z \in 2^A : R_{x,Y} \wedge (\forall y \in Y : R_{y,Z}) \Rightarrow R_{x,Z} \\
\iff &\forall x \in A : \forall Y, Z \in 2^A : R_{x,Y} \wedge (\forall y \in A : y \in Y \Rightarrow R_{y,Z}) \Rightarrow R_{x,Z} \\
\iff &\forall x \in A : \forall Y, Z \in 2^A : R_{x,Y} \wedge (\forall y \in A : \mathsf{E}_{y,Y} \Rightarrow R_{y,Z}) \Rightarrow R_{x,Z} \\
\iff &\forall x \in A : \forall Y, Z \in 2^A : R_{x,Y} \wedge (\mathsf{E} \setminus R)_{Y,Z} \Rightarrow R_{x,Z} \\
\iff &\forall x \in A : \forall Z \in 2^A : (\exists Y \in 2^A : R_{x,Y} \wedge (\mathsf{E} \setminus R)_{Y,Z}) \Rightarrow R_{x,Z} \\
\iff &\forall x \in A : \forall Z \in 2^A : (R(\mathsf{E} \setminus R))_{x,Z} \Rightarrow R_{x,Z} \\
\iff &R(\mathsf{E} \setminus R) \subseteq R \\
\iff &R;R \subseteq R
\end{aligned}$$

Again Figure 1 shows the claim. □

Taken together, the axioms (K_1) to (K_3) of Aumann are equivalent to multirelations being reflexive, up-closed and transitive (or even idempotent, since reflexive implies dense). Finally, we investigate axiom (K_4). Here we obtain the following results.

Theorem 12. *Multirelations satisfying* (K_4) *are* $\cup$*-distributive. An up-closed multirelation satisfies* (K_4) *if and only if it is* $\cup$*-distributive.*

Proof. Let $R : A \leftrightarrow 2^B$ be a multirelation such that axiom (K_4) holds. Because of inclusion (8) of Theorem 2 we only have to show $R;(P \cup Q) \subseteq R;P \cup R;Q$ for all multirelations $P : B \leftrightarrow 2^C$ and $Q : B \leftrightarrow 2^C$ to verify the first claim. To this end let $x \in A$ and $X \in 2^C$ such that $(R;(P \cup Q))_{x,X}$. Then there exists $W \in 2^B$ such that $R_{x,W}$ and for all $y \in W$ also $P_{y,X}$ or $Q_{y,X}$. We define two sets $Y, Z \in 2^B$ as subsets of W as follows:

$$Y := \{y \in W \mid P_{y,X}\} \qquad Z := \{y \in W \mid Q_{y,X}\}$$

Then we get $W = Y \cup Z$. Hence, we have $R_{x,Y}$ or $R_{x,Z}$ by the assumption that (K_4) holds. In the first case this shows $(R;P)_{x,X}$, since $P_{y,X}$ for all $y \in Y$, and in the second case $(R;Q)_{x,X}$.

To prove the second claim, assume that R is up-closed and $\cup$-distributive and let $x \in A$ and $Y, Z \in 2^B$ be given. First, suppose $R_{x,Y\cup Z}$. We define the up-closed multirelations $P : A \leftrightarrow 2^B$ and $Q : A \leftrightarrow 2^B$ as follows:

$$P := \{(x, X) \in R \mid x \in X \cap Y\} \qquad Q := \{(x, X) \in R \mid x \in X \cap Z\}$$

Then we have $P_{y,Y\cup Z}$ for all $y \in Y$ and also $Q_{y,Y\cup Z}$ for all $y \in Z$. This leads to $(P \cup Q)_{y,Y\cup Z}$ for all $y \in Y \cup Z$, which gives $(R;(P \cup Q))_{x,Y\cup Z}$. By the assumption $(R;P)_{x,Y\cup Z}$ or $(R;Q)_{x,Y\cup Z}$ holds. In the first case there exists $W \in 2^B$ such that $R_{x,W}$ and $P_{y,Y\cup Z}$ for all $y \in W$. The definition of P implies that $y \in Y$ for all $y \in W$, thus $W \subseteq Y$. Since R is up-closed, this shows $R_{x,Y}$. In the second case, $R_{x,Z}$ follows analogously using the definition of Q. Altogether, $R_{x,Y\cup Z}$ implies $R_{x,Y}$ or $R_{x,Z}$. To prove the converse implication, suppose $R_{x,Y}$ or $R_{x,Z}$. In both cases we then get $R_{x,Y\cup Z}$ since R is up-closed. □

Extended to arbitrary non-empty unions, axiom (K_4) is called 'additive' in [23], which also states that additive up-closed multirelations are $\cup$-distributive.

Finally we consider the dual property of axiom (K_4), that is, the following logical formula for a given multirelation $R : A \leftrightarrow 2^B$:

$$(K_4') \quad \forall x \in A : \forall Y, Z \in 2^B : R_{x,Y} \wedge R_{x,Z} \Leftrightarrow R_{x,Y\cap Z}$$

Extended to arbitrary non-empty unions, this is called 'multiplicative' in [24], which also states that multiplicative up-closed multirelations are $\cap$-distributive. Similarly to the proof of Theorem 12 the following result can be shown.

Theorem 13. *Multirelations satisfying* (K_4') *are* $\cap$*-distributive. An up-closed multirelation satisfies* (K_4') *if and only if it is* $\cap$*-distributive.*

9 Conclusion

In this paper we investigated multirelations using relation algebras and more general algebraic structures. In particular, we considered various properties of multirelations that have been used in applications and we studied transitive closures, closure properties and Aumann contacts.

In Figure 1 we also mentioned vectors and tests and we will close with some remarks concerning these notions. Relational tests are used to represent sets. Such a test is a relation $p : A \leftrightarrow A$ with $p \subseteq \mathsf{I}$ and represents the set $\{x \in A \mid p_{x,x}\}$. A straight-forward generalisation to multirelations would take multirelations which are contained in the membership relation $\mathsf{E} : A \leftrightarrow 2^A$ as tests. But there are too many such multirelations, most of which are not up-closed. This would lead to problems, as tests are frequently used in combination with multirelational composition to restrict a computation to a set of starting states. As a solution, [14] defines multirelational tests as intersections of multirelational vectors in the sense of Figure 1 with membership relations. Hence, a multirelation $R : A \leftrightarrow 2^A$ is a *test* if $R = R;\mathsf{T} \cap \mathsf{E}$, as stated in Figure 1. Using this definition it can be shown that $P : A \leftrightarrow 2^A$ is a multirelational test if and only if there exists a relational test $p : A \leftrightarrow A$ such that $P = p\mathsf{E}$. Furthermore, as for relational tests, composition and intersection of tests coincide, that is, for multirelational tests P and Q we have $P;Q = P \cap Q$.

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