A Relation-Algebraic Approach to Multirelations and Predicate Transformers

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Abstract. The correspondence between up-closed multirelations and isotone predicate transformers is well known. Less known is that multirelations have also been used for modelling topological contact, not only computations. We investigate how properties from these two lines of research translate to predicate transformers. To this end, we express the correspondence of multirelations and predicate transformers using relation algebras. It turns out to be similar to the correspondence between contact relations and closure operations. Many results generalise from up-closed to arbitrary multirelations.

1 Introduction

Predicate transformers have been used for defining the semantics of programs and constructing correct programs for a long time; see, for example, [9,12]. A multirelational representation of isotone predicate transformers has been given in [25]. Multirelations – relations between a set and a powerset – have previously been used for defining the semantics of game-based computations and logics [22] as well as to model contact and related notions from topology [1,2].

In the companion paper [5] we have started to bring together the computational and topological lines of research on multirelations. In particular, we consider various properties of multirelations that have been used in these lines of research. We investigate how these properties are related and under which operations they are closed by introducing general algebras. An observation from this work is that being up-closed is just one among many useful properties.

Research about multirelations in program semantics commonly makes the restriction to up-closed multirelations, perhaps due to the corresponding assumption that predicate transformers are isotone. Motivated by our previous work we raise the question whether the investigation can be and should be liberated from the restriction to up-closed multirelations. That it is possible is indicated by both this paper and the companion paper: many results do not require this restriction. That it is desirable is indicated by some applications in programming, notably concurrent dynamic logic [23], which cannot be restricted to up-closed multirelations.

The present paper continues the programme started in the companion paper. Our aim is to investigate the correspondence between multirelations and predicate transformers generally and to use it for translating multirelational operations and properties to predicate transformers. Our contributions are mainly presented in Section 4 after a basic discussion of relations, multirelations, contact relations and predicate transformers in Sections 2 and 3. They are (1) a relation-algebraic description of an order isomorphism between multirelations and predicate transformers in Section 4.1, (2) a relation-algebraic translation of multirelational operations to predicate transformers in Section 4.2, (3) an alternative composition of multirelations in Section 4.3, and (4) relation-algebraic and logical translations of multirelational properties to predicate transformers in Section 4.4. Some properties from the topological line of research might be less well understood on the computational side. We show an example that uses such a property to weaken the assumptions for a program transformation, though the focus of this paper is a foundational investigation.

Among related work we chiefly mention [18], which uses power allegories for a categorical approach to up-closed multirelations. The article [20] studies the relationship of up-closed multirelations, predicate transformers and other models for representing higher-order functions. The present paper uses relation algebras and investigates arbitrary multirelations. Other related work is discussed throughout the paper.

2 Preliminaries

In this section we present facts of relation algebras needed in the remainder of the paper. For more details on relations and relation algebras, see [28,30,31].

2.1 Relation Algebras

Following the Z notation, we write $R : A \leftrightarrow B$ if R is a (typed, binary) relation with source A and target B, that is, a subset of the Cartesian product $A \times B$. If the sets A and B of the *type* $A \leftrightarrow B$ of R are finite we may consider R as a Boolean matrix with $|A|$ rows and $|B|$ columns. This interpretation is well suited for many purposes. Therefore, we use matrix notation in this paper and write $R_{x,y}$ instead of $(x,y) \in R$ or $x\,R\,y$.

We assume the reader is familiar with the basic operations on relations, namely R^{c} (transposition, converse), $\overline{R}$ (complement, negation), $R \cup S$ (union, join), $R \cap S$ (intersection, meet), RS (composition, product), the predicates indicating $R \subseteq S$ (inclusion) and $R = S$ (equality), and the special relations O (empty relation), T (universal relation) and I (identity relation).

We use $^{-}$, $\cup$, $\cap$ and $\subseteq$ for arbitrary sets, not just relations. With these set operations, the subset order and the constants $\mathsf{O} : A \leftrightarrow B$ and $\mathsf{T} : A \leftrightarrow B$, the set of relations of a type $A \leftrightarrow B$ forms a complete Boolean lattice. Well-known rules involving transposition and composition are, for instance, $R^{\mathsf{cc}} = R$, $\overline{R^{\mathsf{c}}} = \overline{R}^{\mathsf{c}}$, $(R \cup S)^{\mathsf{c}} = R^{\mathsf{c}} \cup S^{\mathsf{c}}$, $(R \cap S)^{\mathsf{c}} = R^{\mathsf{c}} \cap S^{\mathsf{c}}$, $(RS)^{\mathsf{c}} = S^{\mathsf{c}} R^{\mathsf{c}}$, $Q(R \cup S) = QR \cup QS$

and $Q(R \cap S) \subseteq QR \cap QS$. Moreover, transposition is $\subseteq$-isotone and union, intersection and composition are $\subseteq$-isotone in both arguments.

The theoretical framework for these rules and many others is that of an (axiomatic, heterogeneous) relation algebra in the sense of [28,30], which generalises the original homogeneous approach of [31]. As constants and operations of this algebraic structure we have those of concrete (that is, set-theoretic) relations. The axioms of a relation algebra are those of a complete Boolean lattice for the set operations, the associativity of composition, the neutrality of identity relations for composition, and the equivalences

$$QR \subseteq S \iff Q^{\mathsf{c}}\overline{S} \subseteq \overline{R} \iff \overline{S}R^{\mathsf{c}} \subseteq \overline{Q}$$

for all relations $Q : A \leftrightarrow B$, $R : B \leftrightarrow C$ and $S : A \leftrightarrow C$ or – equivalently – the so-called Dedekind rule

$$QR \cap S \subseteq (Q \cap SR^{\mathsf{c}})(R \cap Q^{\mathsf{c}}S).$$

We assume all relation-algebraic expressions and formulae to be well-typed and suppress type information if appropriate. Many relation-algebraic notions are also available in the settings of categories and allegories [10]. In particular, related category-theoretic axiomatisations are used in [8] for program development.

Residuals are the greatest solutions of certain inclusions. They appear as weakest prespecifications in [15] and as factors in [4]. The *left residual* of the relation $S : A \leftrightarrow C$ over $R : B \leftrightarrow C$, in symbols $S/R : A \leftrightarrow B$, is the greatest relation $X : A \leftrightarrow B$ such that $XR \subseteq S$. So, we have the Galois connection $XR \subseteq S$ if and only if $X \subseteq S/R$, for all relations $X : A \leftrightarrow B$. Similarly, the *right residual* of $S : A \leftrightarrow C$ over $R : A \leftrightarrow B$, in symbols $R \setminus S : B \leftrightarrow C$, is the greatest relation $X : B \leftrightarrow C$ such that $RX \subseteq S$. This implies that $RX \subseteq S$ if and only if $X \subseteq R \setminus S$, for all relations $X : B \leftrightarrow C$.

We will also need relations which are left and right residuals simultaneously. The *symmetric quotient* $\mathsf{syq}(R, S) : B \leftrightarrow C$ of two relations $R : A \leftrightarrow B$ and $S : A \leftrightarrow C$ is defined as the greatest relation $X : B \leftrightarrow C$ such that $RX \subseteq S$ and $XS^{\mathsf{c}} \subseteq R^{\mathsf{c}}$. In terms of the basic operations we have for all relations R and S of appropriate type the following descriptions:

$$S/R = \overline{\overline{S}R^{\mathsf{c}}} \qquad R \setminus S = \overline{R^{\mathsf{c}}\overline{S}} \qquad \mathsf{syq}(R, S) = (R \setminus S) \cap (R^{\mathsf{c}}/S^{\mathsf{c}})$$

Further properties of the two residuals and the symmetric quotient we will use in this paper can be found in [28,30]; for example, $\mathsf{syq}(R, S)^{\mathsf{c}} = \mathsf{syq}(S, R)$. We assume that the unary relation-algebraic operations have the highest precedence, then composition follows and its precedence is higher than that of union, intersection and the residuals, all of which have the same precedence.

2.2 Mappings and Predicate Transformers

The basic operations and constants mentioned in Section 2.1 can be used for defining specific classes of relations in a purely algebraic way. In the following we introduce the classes that will be used in the remainder of this paper. For more details we refer again to [28,30].

A relation $R : A \leftrightarrow B$ is *univalent* if $R^{\mathsf{c}}R \subseteq \mathsf{I}$, and *total* if $R\mathsf{T} = \mathsf{T}$. The latter equation is equivalent to the inclusion $\mathsf{I} \subseteq RR^{\mathsf{c}}$. We call R a *mapping* (from A to B) if it is univalent and total. In the case of mappings we use small letters and the common type annotation. Hence, if we write $f : A \to B$, then f is a relation of type $A \leftrightarrow B$ that is univalent and total. In pointwise arguments we also use $f(x)$ for function application; for example, $f(x) = y$ if and only if $f_{x,y}$.

For a univalent relation $R : A \leftrightarrow B$ we have $R\overline{S} \subseteq \overline{RS}$ and for a total relation $R : A \leftrightarrow B$ we have $R\overline{S} \supseteq \overline{RS}$, for all relations $S : B \leftrightarrow C$. Hence, for a mapping $f : A \to B$ we get $f\overline{S} = \overline{fS}$, for all relations $S : B \leftrightarrow C$. Moreover, the shunting property $Rf \subseteq S$ if and only if $R \subseteq Sf^{\mathsf{c}}$ holds for any mapping $f : B \to C$ and relations $R : A \leftrightarrow B$ and $S : A \leftrightarrow C$.

The relation $R : A \leftrightarrow B$ is called *injective* if $R^{\mathsf{c}} : B \leftrightarrow A$ is univalent, *surjective* if R^{c} is total and *bijective* if R^{c} is a mapping. Hence $\overline{S}R \subseteq \overline{SR}$ if R is injective, $\overline{S}R \supseteq \overline{SR}$ if R is surjective, and $\overline{S}R = \overline{SR}$ if R is bijective, for all relations $S : C \leftrightarrow A$. The converse shunting property is $fR \subseteq S$ if and only if $R \subseteq f^{\mathsf{c}}S$ for bijective $f : A \to B$ and relations $R : B \leftrightarrow C$ and $S : A \leftrightarrow C$.

If the relation Q is univalent, the subdistributivity $Q(R \cap S) \subseteq QR \cap QS$ becomes an equality $Q(R \cap S) = QR \cap QS$. A consequence of $f\overline{S} = \overline{fS}$ and $f(R \cap S) = fR \cap fS$ for a mapping f is the following result.

Lemma 1. *For all relations $R : A \leftrightarrow B$ and $S : A \leftrightarrow C$ and all mappings $f : D \to B$ we have $\mathsf{syq}(Rf^{\mathsf{c}}, S) = f\mathsf{syq}(R, S)$ and for all mappings $g : D \to C$ we have $R \setminus Sg^{\mathsf{c}} = (R \setminus S)g^{\mathsf{c}}$.*

Proof. The first claim is [6, Theorem 3.3]. We obtain the second claim by

$$R \setminus Sg^{\mathsf{c}} = \overline{R^{\mathsf{c}}\overline{Sg^{\mathsf{c}}}} = \overline{R^{\mathsf{c}}\overline{S}g^{\mathsf{c}}} = \overline{R^{\mathsf{c}}\overline{S}}g^{\mathsf{c}} = (R \setminus S)g^{\mathsf{c}},$$

using that g is a mapping in the second and third steps. □

Following general terminology, a *predicate transformer* (as introduced in [9] for weakest-precondition semantics) is a function that maps a predicate on the state space A of a program to a predicate on the same space. If we consider a predicate on A as a subset of A, that is, as an element of the powerset 2^A, then a predicate transformer is just a function from 2^A to 2^A. Therefore, allowing different state spaces, we call a mapping $f : 2^B \to 2^A$ in the relational sense a (relational) predicate transformer.

2.3 Relation-Algebraic Specification of Set-Theoretic Constructions

Besides empty relations, universal relations and identity relations, we need further specific relations for fundamental set-theoretic constructions. They are introduced in the following.

Let A be a set. The *membership relation* $\mathsf{E} : A \leftrightarrow 2^A$ is the relation-level equivalent to the set-theoretic predicate "$\in$". Hence, we have $\mathsf{E}_{x,Y}$ if and only if $x \in Y$, for all $x \in A$ and $Y \in 2^A$. With the help of E, a right residual and a symmetric quotient we can introduce two relations on 2^A as follows:

$$\mathsf{S} := \mathsf{E} \setminus \mathsf{E} : 2^A \leftrightarrow 2^A \qquad\qquad \mathsf{C} := \mathsf{syq}(\overline{\mathsf{E}}, \mathsf{E}) : 2^A \leftrightarrow 2^A$$

A little pointwise calculation shows that $\mathsf{S}_{X,Y}$ if and only if $X \subseteq Y$ and $\mathsf{C}_{X,Y}$ if and only if $Y = \overline{X}$, for all $X, Y \in 2^A$, where $\overline{X}$ is the complement of the set X relative to its superset A. Therefore, we call S a *subset relation* and C a *set complement relation*. We use $\mathsf{C}_A, \mathsf{S}_A : 2^A \leftrightarrow 2^A$ and $\mathsf{C}_B, \mathsf{S}_B : 2^B \leftrightarrow 2^B$ to clarify the type if necessary.

We next specify the two binary operations of meet and join of sets as two relations $\mathsf{M} : 2^A \times 2^A \leftrightarrow 2^A$ (*meet relation*) and $\mathsf{J} : 2^A \times 2^A \leftrightarrow 2^A$ (*join relation*) such that $\mathsf{M}_{(X,Y),Z}$ if and only if $X \cap Y = Z$ and $\mathsf{J}_{(X,Y),Z}$ if and only if $X \cup Y = Z$, for all sets $X, Y, Z \in 2^A$. To this end, besides the membership relation $\mathsf{E} : A \leftrightarrow 2^A$ we need the two projections of the direct product $2^A \times 2^A$ as relation-algebraic mappings $p : 2^A \times 2^A \to 2^A$ and $r : 2^A \times 2^A \to 2^A$. They satisfy $p_{(X,Y),Z}$ if and only if $X = Z$ and $r_{(X,Y),Z}$ if and only if $Y = Z$, for all sets $X, Y, Z \in 2^A$. This allows us to derive the following relation-algebraic specifications:

$$\mathsf{M} := \mathsf{syq}([\mathsf{E}, \mathsf{E}]\!], \mathsf{E}) : 2^A \times 2^A \leftrightarrow 2^A \qquad\qquad \mathsf{J} := \mathsf{syq}(\overline{[\overline{\mathsf{E}}, \overline{\mathsf{E}}]\!]}, \mathsf{E}) : 2^A \times 2^A \leftrightarrow 2^A$$

In these two definitions $[\cdot, \cdot]\!]$ denotes the *right pairing* operation of the direct product $2^A \times 2^A$, also known as fork or tupling operation. Using the projection mappings $p, r : 2^A \times 2^A \to 2^A$, the right pairing of relations $R : B \leftrightarrow 2^A$ and $S : B \leftrightarrow 2^A$ is defined as

$$[R, S]\!] := Rp^{\mathsf{c}} \cap Sr^{\mathsf{c}} : B \leftrightarrow 2^A \times 2^A.$$

As a consequence we have $[\mathsf{E}, \mathsf{E}]\!]_{x,(X,Y)}$ if and only if $x \in X$ and $x \in Y$, for all $x \in A$ and $X, Y \in 2^A$. From this, a little pointwise calculation yields

$$\mathsf{syq}([\mathsf{E}, \mathsf{E}]\!], \mathsf{E})_{(X,Y),Z} \iff (\forall x \in A : x \in X \wedge x \in Y \Leftrightarrow x \in Z) \iff X \cap Y = Z$$

for all sets $X, Y, Z \in 2^A$, that is, $\mathsf{M}_{(X,Y),Z}$ if and only if $X \cap Y = Z$. Similarly it can be verified that the relation J specifies the join of two sets from 2^A.

The specifications of set-theoretic constructions via the relations S, C, M and J are not yet purely relation-algebraic, since they are still based on the pointwise definitions of membership relations and projection mappings. However, both can be specified with purely relation-algebraic means in a monomorphic manner [28,30]. For the membership relation $\mathsf{E} : A \leftrightarrow 2^A$ we have

$$\mathsf{syq}(\mathsf{E}, \mathsf{E}) = \mathsf{I} \qquad\qquad \forall R : \mathsf{T}\,\mathsf{syq}(\mathsf{E}, R) = \mathsf{T}$$

as a second-order axiomatisation. From this we get, for example, $\mathsf{E}\,\mathsf{syq}(\mathsf{E}, R) = R$. The property $\mathsf{syq}(\mathsf{E}, \mathsf{E}) = \mathsf{I}$ captures set equality by mutual inclusion given that $\mathsf{syq}(\mathsf{E}, \mathsf{E}) = \mathsf{S} \cap \mathsf{S}^{\mathsf{c}}$. First-order axioms for the projection mappings are

$$p^{\mathsf{c}}p = \mathsf{I} \qquad r^{\mathsf{c}}r = \mathsf{I} \qquad pp^{\mathsf{c}} \cap rr^{\mathsf{c}} = \mathsf{I} \qquad p^{\mathsf{c}}r = \mathsf{T}.$$

From these axioms we immediately get that the relations p and r are in fact mappings and that their transposes p^{c} and r^{c} are total. Hence, p and r are surjective mappings which is characteristic for projections. The axioms for projections work for general products of type $A \times B$, not just for the instance $2^A \times 2^A$ needed above; similar axioms of biproducts have been used in the context of linear algebra [17].

In the following series of three lemmas we collect further properties needed in the remainder of this paper. The following properties of S are shown in [29].

Lemma 2. *Each subset relation* S *is reflexive* ($\mathsf{I} \subseteq \mathsf{S}$), *antisymmetric* ($\mathsf{S} \cap \mathsf{S}^{\mathsf{c}} \subseteq \mathsf{I}$) *and transitive* ($\mathsf{SS} \subseteq \mathsf{S}$), *that is, a partial order relation.*

Basic laws of symmetric quotients [30, Theorems 4.4.1 and 4.4.3] and the second axiom of the membership relation $\mathsf{E} : A \leftrightarrow 2^A$ yield that $\mathsf{syq}(R, \mathsf{E}) : B \leftrightarrow 2^A$ is a mapping for all relations $R : A \leftrightarrow B$. Therefore we may write $\mathsf{C} : 2^A \to 2^A$ and $\mathsf{M}, \mathsf{J} : 2^A \times 2^A \to 2^A$. For each set complement mapping we furthermore have the following result that, in combination with the mapping property, shows that C is bijective. It is proved in [27].

Lemma 3. *For each set complement mapping* C *we have* $\mathsf{C} = \mathsf{C}^{\mathsf{c}}$.

We formulate the properties of the next lemma only for the projection mappings $p, r : 2^A \times 2^A \to 2^A$ of the direct product $2^A \times 2^A$ and two mappings $f, g : 2^A \to 2^A$, but they obviously hold for arbitrary projections and mappings of appropriate type.

Lemma 4. *Let* $f, g : 2^B \to 2^A$ *be mappings. Then*

(1) $[f, g]\!] p = f$ *and* $[f, g]\!] r = g$,
(2) $[f, g]\!]$ *is a mapping,*
(3) $[\mathsf{E}, \mathsf{E}]\!] [f, g]\!]^{\mathsf{c}} = \mathsf{E} f^{\mathsf{c}} \cap \mathsf{E} g^{\mathsf{c}}$.

Proof. (1) The first equality holds because of the following calculation, which uses two axioms of the projection mappings and the Dedekind rule; the second equality is proved similarly:

$$f = f \cap g\mathsf{T} = f \cap g r^{\mathsf{c}} p \subseteq (g r^{\mathsf{c}} \cap f p^{\mathsf{c}}) p = [f, g]\!] p = (f p^{\mathsf{c}} \cap g r^{\mathsf{c}}) p \subseteq f p^{\mathsf{c}} p = f.$$

(2) Using an axiom of the projection mappings, $[f, g]\!]$ is univalent since

$$[f, g]\!]^{\mathsf{c}} [f, g]\!] = (p f^{\mathsf{c}} \cap r g^{\mathsf{c}})(f p^{\mathsf{c}} \cap g r^{\mathsf{c}}) \subseteq p f^{\mathsf{c}} f p^{\mathsf{c}} \cap r g^{\mathsf{c}} g r^{\mathsf{c}} \subseteq p p^{\mathsf{c}} \cap r r^{\mathsf{c}} = \mathsf{I}.$$

Using the first equation of (1) and that the projection p is total, $[f, g]\!]$ is also total because

$$[f, g]\!] \mathsf{T} = [f, g]\!] p \mathsf{T} = f \mathsf{T} = \mathsf{T}.$$

(3) The claim is a special case of [7, Theorem 2.7]. □

3 Relations and Multirelations

In this section we first recall the basic definitions, operations and properties of multirelations. We also present special kinds of multirelations used to model topological contact. Then we show how multirelational composition and the dual can be expressed in terms of relation algebras and present some fundamental properties. For more details we refer to [5,13].

3.1 Multirelations

A *multirelation* (as introduced in [22,25]) is a relation in the sense of Section 2.1 with the additional property that the target is a powerset. So, for sets A and B a multirelation $R : A \leftrightarrow 2^B$ relates an element of A with a subset of B. The Boolean operations *union* $R \cup S$, *intersection* $R \cap S$ and *complement* $\overline{R}$ apply to multirelations R and S (of the same type) as to general relations. Particular multirelations are the *empty multirelations* $\mathsf{O} : A \leftrightarrow 2^B$, the *universal multirelations* $\mathsf{T} : A \leftrightarrow 2^B$ and the *membership multirelations* $\mathsf{E} : A \leftrightarrow 2^A$. Hence $\mathsf{O} = \emptyset$ and $\mathsf{T} = A \times 2^B$. The *composition* of the multirelations $Q : A \leftrightarrow 2^B$ and $R : B \leftrightarrow 2^C$ is the multirelation $Q;R : A \leftrightarrow 2^C$ pointwise defined by

$$(Q;R)_{x,Z} \iff \exists Y \in 2^B : Q_{x,Y} \wedge \forall y \in Y : R_{y,Z},$$

for all $x \in A$ and $Z \in 2^C$. Being relations, multirelations also can be transposed, but the result is not a multirelation whence this plays no role. Instead a dual operation is used. The *dual* of a multirelation $R : A \leftrightarrow 2^B$ is the multirelation $R^{\mathsf{d}} : A \leftrightarrow 2^B$ pointwise defined by

$${R^{\mathsf{d}}}_{x,Y} \iff \neg R_{x,\overline{Y}},$$

for all $x \in A$ and $Y \in 2^B$, where $\overline{Y}$ denotes the complement of the set Y relative to its superset B. We assume that the unary operations complement and dual have the highest precedence, then composition follows which has higher precedence than union and intersection. Finally, a multirelation $R : A \leftrightarrow 2^B$ is *up-closed* if

$$R_{x,Y} \wedge Y \subseteq Z \implies R_{x,Z}$$

for all $x \in A$ and $Y, Z \in 2^B$. This means that if an element of A is related to a set Y it also has to be related to all supersets of Y.

3.2 Contact Relations

Already before applications in program semantics, multirelations of type $A \leftrightarrow 2^A$ were used in [1] for modelling contact in order to introduce topological concepts. In particular, the following axioms for $R : A \leftrightarrow 2^A$ were considered:

(K_0) $\neg\exists x \in A : R_{x,\emptyset}$
(K_1) $\forall x \in A : R_{x,\{x\}}$
(K_2) $\forall x \in A : \forall Y, Z \in 2^A : R_{x,Y} \wedge Y \subseteq Z \Rightarrow R_{x,Z}$
(K_3) $\forall x \in A : \forall Y, Z \in 2^A : R_{x,Y} \wedge (\forall y \in Y : R_{y,Z}) \Rightarrow R_{x,Z}$
(K_4) $\forall x \in A : \forall Y, Z \in 2^A : R_{x,Y \cup Z} \Leftrightarrow R_{x,Y} \vee R_{x,Z}$

Multirelations satisfying axioms (K_1) to (K_3) are called "contact relations" and those satisfying axioms (K_0) to (K_4) are called "topological contact relations" in [1]. Algebraic characterisations of these logical formulas are derived in [5] and will be discussed in Section 4.4. Axiom (K_2) states that R is up-closed.

3.3 Modelling Multirelational Composition and the Dual

Since multirelations are relations, with respect to the Boolean operations the set of multirelations of a given type $A \leftrightarrow 2^B$ forms a complete Boolean lattice with $\mathsf{O} : A \leftrightarrow 2^B$ as least element and $\mathsf{T} : A \leftrightarrow 2^B$ as greatest element of the subset order. To get algebraic laws for multirelational composition and dual we express these operations and the property of being up-closed in terms of relation-algebraic operations and constants, namely composition, complement, right residual, membership relation E, set complement relation C and subset relation S.

Theorem 5. *Let $Q : A \leftrightarrow 2^B$ and $R : B \leftrightarrow 2^C$ be multirelations. Then*

$$\text{(1) } Q\,;R = Q(\mathsf{E} \setminus R) \qquad\qquad \text{(2) } Q^{\mathsf{d}} = \overline{Q}\mathsf{C} = \overline{Q\mathsf{C}}.$$

Moreover, Q is up-closed if and only if $Q = Q\mathsf{S}$, which is equivalent to $Q\mathsf{S} \subseteq Q$.

The algebraic laws of the multirelational operations become more diversified if multirelational composition is taken into account. The familiar relation-algebraic laws that composition distributes over union and has the empty relation as a zero only hold from one side. Other laws of relation algebras, namely, that composition is associative and has the identity relation as a neutral element, hold for up-closed multirelations (with I replaced by E) but need to be weakened in the general case. On the other hand, composition remains $\subseteq$-isotone. These and related properties are summarised in the following result.

Theorem 6. *For all multirelations P, Q and R we have*

$$\text{(1) } \mathsf{O}\,;R = \mathsf{O} \qquad \text{(2) } \mathsf{E}\,;R = R \qquad \text{(3) } \mathsf{T}\,;R = \mathsf{T} \qquad \text{(4) } R \subseteq R\,;\mathsf{E},$$

where in (4) equality holds if and only if R is up-closed, and also

$$\text{(5) } (P \cup Q)\,;R = P\,;R \cup Q\,;R, \qquad \text{(6) } (P \cap Q)\,;R \subseteq P\,;R \cap Q\,;R,$$

where in (6) equality holds if P and Q are up-closed, and also

$$\text{(7) } (P\,;Q)\,;R \subseteq P\,;(Q\,;R),$$

where in (7) equality holds if Q is up-closed, and finally

$$\text{(8) } P\,;Q \cup P\,;R \subseteq P\,;(Q \cup R) \qquad \text{(9) } P\,;(Q \cap R) \subseteq P\,;Q \cap P\,;R.$$

Finally, we consider algebraic laws of the dual. This operation reverses the lattice order and distributes over composition of up-closed multirelations. Again this needs to be weakened in the general case. These and further properties are summarised in the following result.

Theorem 7. *For all multirelations Q and R we have*

(1) $\mathsf{O}^{\mathsf{d}} = \mathsf{T}$	*(4)* $R^{\mathsf{dd}} = R$	
(2) $\mathsf{E}^{\mathsf{d}} = \mathsf{E}$	*(5)* $(Q \cup R)^{\mathsf{d}} = Q^{\mathsf{d}} \cap R^{\mathsf{d}}$	*(7)* $(Q\,;R)^{\mathsf{d}} \subseteq Q^{\mathsf{d}}\,;R^{\mathsf{d}}$
(3) $\mathsf{T}^{\mathsf{d}} = \mathsf{O}$	*(6)* $(Q \cap R)^{\mathsf{d}} = Q^{\mathsf{d}} \cup R^{\mathsf{d}}$	*(8)* $(Q\,;R)^{\mathsf{d}} = (Q\,;\mathsf{E})^{\mathsf{d}}\,;R^{\mathsf{d}}$,

where in (7) equality holds if Q is up-closed.

For proofs of Theorems 5–7 we refer to [5,13,18,28].

4 Multirelations and Predicate Transformers

A one-to-one correspondence between contact relations and closure operations is given in [1] and treated relation-algebraically in [29]. In this section we first generalise this correspondence to one between multirelations and predicate transformers. Based on this, we show how to translate operations and properties between these structures. We come back to contact relations in Section 4.4.

4.1 Connecting Multirelations and Predicate Transformers

Let $R : A \leftrightarrow 2^B$ be a multirelation. By forming the symmetric quotient with the membership multirelation $\mathsf{E} : A \leftrightarrow 2^A$ we obtain

$$\Psi(R) := \mathsf{syq}(R, \mathsf{E}) : 2^B \leftrightarrow 2^A.$$

From Section 2.3 we know that the specific symmetric quotient construction $\mathsf{syq}(R, \mathsf{E})$ is always a mapping in the sense of Section 2.2. As a consequence, we are allowed to write $\Psi(R) : 2^B \to 2^A$ for typing the relation $\Psi(R)$ and the mapping $\Psi(R)$ becomes a predicate transformer in the sense of Section 2.2. Conversely, let $f : 2^B \to 2^A$ be a predicate transformer. Using the membership multirelation $\mathsf{E} : A \leftrightarrow 2^A$, by typing reasons we get

$$\Phi(f) := \mathsf{E}f^{\mathsf{c}} : A \leftrightarrow 2^B,$$

whence the relation $\Phi(f)$ becomes a multirelation. Set-theoretic definitions of the two functions (in the usual mathematical sense) Ψ and Φ between the set of relations of type $A \leftrightarrow 2^B$ and the set of functions (again in the usual mathematical sense) from 2^B to 2^A have already been given in [1]. The above relation-algebraic specifications of Ψ and Φ come from [29]. For up-closed multirelations and isotone predicate transformers they are expressed using the power transpose of power allegories in [18]. In [8] the power transpose Λ is characterised by the universal property $\Lambda R = f$ if and only if $R = f\mathsf{E}^{\mathsf{c}}$ and it is shown that $\Lambda R = \mathsf{syq}(R^{\mathsf{c}}, \mathsf{E})$. Hence $\Psi(R) = \Lambda(R^{\mathsf{c}})$ holds, and $\Phi(f) = \mathsf{E}f^{\mathsf{c}}$ also follows from the universal property. The following result appears as [29, Corollary 4.2 and Theorem 4.4] and, for up-closed multirelations, as [18, Lemma 6.4].

Theorem 8. *The functions Ψ and Φ are mutually inverse and, thus, constitute a one-to-one correspondence between the multirelations of type $A \leftrightarrow 2^B$ and the mappings of type $2^B \to 2^A$. They are order isomorphisms with respect to the inclusion of multirelations and the pointwise order of mappings.*

4.2 Translating Operations

By means of the one-to-one correspondence of Section 4.1 we are able to translate the operations of union, intersection, composition, dual and complement from multirelations to predicate transformers and to do the same for the three constant multirelations and the inclusion order. The corresponding definitions are as follows; we use the notation of [24].

Definition 9. *Given predicate transformers $f, g : 2^B \to 2^A$ and $h : 2^C \to 2^B$ we define the operations and constants*

$$\begin{array}{rll} \textit{union} & f \sqcup g := \Psi(\Phi(f) \cup \Phi(g)) : 2^B \to 2^A & \bot := \Psi(\mathsf{O}) : 2^B \to 2^A \\ \textit{intersection} & f \sqcap g := \Psi(\Phi(f) \cap \Phi(g)) : 2^B \to 2^A & \top := \Psi(\mathsf{T}) : 2^B \to 2^A \\ \textit{composition} & f \circ h := \Psi(\Phi(f) ; \Phi(h)) \ : 2^C \to 2^A & 1 := \Psi(\mathsf{E}) : 2^A \to 2^A \\ \textit{dual} & f^\circ := \Psi(\Phi(f)^{\mathsf{d}}) \quad : 2^B \to 2^A & \\ \textit{complement} & f^- := \Psi(\overline{\Phi(f)}) \quad : 2^B \to 2^A . & \end{array}$$

Moreover, we define the relation $\sqsubseteq$ on the set of predicate transformers of type $2^B \to 2^A$ by $f \sqsubseteq g$ if and only if $\Phi(f) \subseteq \Phi(g)$, for all $f, g : 2^B \to 2^A$.

The following result elaborates these operations, constants and the relation $\sqsubseteq$ in terms of relation algebras. It expresses the following:

– Multirelational union (intersection, complement) translates to the pointwise union (intersection, complement) of predicate transformers.
– Multirelational dual translates to a mapping that, in the usual mathematical sense, is written as the function $f^\circ(X) = \overline{f(\overline{X})}$.
– The empty (universal) multirelation translates to the predicate transformer which maps everything to the empty (full) set and the membership multirelation translates to the identity predicate transformer.
– The lattice order on the set of multirelations of type $A \leftrightarrow 2^B$ translates to the pointwise order on the set of predicate transformers of type $2^B \to 2^A$, which is induced by the inclusion order of the target 2^A.

The pointwise interpretations are as expected from [24]. Note that a restriction to isotone predicate transformers is not required.

Theorem 10. *Let $f, g : 2^B \to 2^A$ be predicate transformers. Then*

(1) $f \sqcup g = [f, g]\mathsf{J}$, *where $\mathsf{J} : 2^A \times 2^A \to 2^A$ is the join mapping,*
(2) $f \sqcap g = [f, g]\mathsf{M}$, *where $\mathsf{M} : 2^A \times 2^A \to 2^A$ is the meet mapping,*
(3) $f^\circ = \mathsf{C}_B f \mathsf{C}_A$ *and* $f^- = f\mathsf{C}_A$, *where $\mathsf{C}_A : 2^A \to 2^A$ and $\mathsf{C}_B : 2^B \to 2^B$ are set complement mappings,*
(4) $\bot = \overline{\mathsf{TE}}$ *and* $\top = \overline{\mathsf{T}\overline{\mathsf{E}}}$, *where $\mathsf{E} : A \leftrightarrow 2^A$ is the membership relation,*
(5) $1 = \mathsf{I}$,
(6) $f \sqsubseteq g$ *if and only if* $f \subseteq g\mathsf{S}^{\mathsf{c}}$, *where $\mathsf{S} : 2^A \leftrightarrow 2^A$ is the subset relation.*

Proof. (1) Using the projection mappings $p, r : 2^A \times 2^A \to 2^A$ of the direct product $2^A \times 2^A$ for the right pairings we obtain:

$$\begin{aligned}
f \sqcup g &= \Psi(\Phi(f) \cup \Phi(g)) \\
&= \Psi(\mathsf{E}f^{\mathsf{c}} \cup \mathsf{E}g^{\mathsf{c}}) \\
&= \mathsf{syq}(\mathsf{E}(f^{\mathsf{c}} \cup g^{\mathsf{c}}), \mathsf{E}) \\
&= \mathsf{syq}(\mathsf{E}(p^{\mathsf{c}}[f,g\rrbracket^{\mathsf{c}} \cup r^{\mathsf{c}}[f,g\rrbracket^{\mathsf{c}}), \mathsf{E}) && \text{Lemma 4 (1)} \\
&= \mathsf{syq}(\mathsf{E}(p^{\mathsf{c}} \cup r^{\mathsf{c}})[f,g\rrbracket^{\mathsf{c}}, \mathsf{E}) \\
&= [f,g\rrbracket\, \mathsf{syq}((\mathsf{E}p^{\mathsf{c}} \cup \mathsf{E}r^{\mathsf{c}}), \mathsf{E}) && \text{Lemma 4 (2) and Lemma 1} \\
&= [f,g\rrbracket\, \mathsf{syq}((\overline{\overline{\mathsf{E}}p^{\mathsf{c}}} \cup \overline{\overline{\mathsf{E}}r^{\mathsf{c}}}), \mathsf{E}) && p^{\mathsf{c}}, r^{\mathsf{c}} \text{ bijective} \\
&= [f,g\rrbracket\, \mathsf{syq}(\overline{\overline{\mathsf{E}}p^{\mathsf{c}} \cap \overline{\mathsf{E}}r^{\mathsf{c}}}, \mathsf{E}) \\
&= [f,g\rrbracket\, \mathsf{syq}(\overline{[\overline{\mathsf{E}}, \overline{\mathsf{E}}\rrbracket}, \mathsf{E}) \\
&= [f,g\rrbracket \mathsf{J}
\end{aligned}$$

(2) The following calculation shows the claim:

$$\begin{aligned}
f \sqcap g &= \Psi(\Phi(f) \cap \Phi(g)) \\
&= \Psi(\mathsf{E}f^{\mathsf{c}} \cap \mathsf{E}g^{\mathsf{c}}) \\
&= \Psi([\mathsf{E},\mathsf{E}\rrbracket[f,g\rrbracket^{\mathsf{c}}) && \text{Lemma 4 (3)} \\
&= \mathsf{syq}([\mathsf{E},\mathsf{E}\rrbracket[f,g\rrbracket^{\mathsf{c}}, \mathsf{E}) \\
&= [f,g\rrbracket\, \mathsf{syq}([\mathsf{E},\mathsf{E}\rrbracket, \mathsf{E}) && \text{Lemma 4 (2) and Lemma 1} \\
&= [f,g\rrbracket \mathsf{M}
\end{aligned}$$

(3) For the dual we proceed as follows; the complement is treated similarly:

$$\begin{aligned}
f^{\circ} &= \Psi(\Phi(f)^{\mathsf{d}}) \\
&= \Psi((\mathsf{E}f^{\mathsf{c}})^{\mathsf{d}}) \\
&= \Psi(\overline{\mathsf{E}f^{\mathsf{c}}}\mathsf{C}_B) && \text{Theorem 5 (2)} \\
&= \mathsf{syq}(\overline{\mathsf{E}f^{\mathsf{c}}}\mathsf{C}_B{}^{\mathsf{c}}, \mathsf{E}) && \text{Lemma 3} \\
&= \mathsf{C}_B\, \mathsf{syq}(\overline{\mathsf{E}}f^{\mathsf{c}}, \mathsf{E}) && \mathsf{C}_B \text{ mapping, Lemma 1, } f \text{ mapping} \\
&= \mathsf{C}_B f\, \mathsf{syq}(\overline{\mathsf{E}}, \mathsf{E}) && f \text{ mapping, Lemma 1} \\
&= \mathsf{C}_B f \mathsf{C}_A
\end{aligned}$$

(4) The two equations are shown by the calculations

$$\begin{aligned}
\bot &= \Psi(\mathsf{O}) = \mathsf{syq}(\mathsf{O},\mathsf{E}) = (\mathsf{O} \setminus \mathsf{E}) \cap (\mathsf{O}^{\mathsf{c}}/\mathsf{E}^{\mathsf{c}}) = \mathsf{T} \cap \overline{\overline{\mathsf{O}}\mathsf{E}} = \overline{\mathsf{T}\mathsf{E}} \\
\top &= \Psi(\mathsf{T}) = \mathsf{syq}(\mathsf{T},\mathsf{E}) = (\mathsf{T} \setminus \mathsf{E}) \cap (\mathsf{T}^{\mathsf{c}}/\mathsf{E}^{\mathsf{c}}) = \overline{\mathsf{T}^{\mathsf{c}}\overline{\mathsf{E}}} \cap \mathsf{T} = \overline{\mathsf{T}\overline{\mathsf{E}}}.
\end{aligned}$$

(5) Using the first axiom of membership relations, a proof of the claim is:

$$1 = \Psi(\mathsf{E}) = \mathsf{syq}(\mathsf{E},\mathsf{E}) = \mathsf{I}$$

(6) To verify this fact we calculate as follows:

$$\begin{aligned}
f \sqsubseteq g &\iff \Phi(f) \subseteq \Phi(g) \\
&\iff \mathsf{E}f^{\mathsf{c}} \subseteq \mathsf{E}g^{\mathsf{c}} \\
&\iff \mathsf{E}f^{\mathsf{c}}g \subseteq \mathsf{E} && g \text{ mapping, shunting} \\
&\iff f^{\mathsf{c}}g \subseteq \mathsf{E} \setminus \mathsf{E} && \text{Galois right residual} \\
&\iff g^{\mathsf{c}}f \subseteq \mathsf{S}^{\mathsf{c}} \\
&\iff f \subseteq g\mathsf{S}^{\mathsf{c}} && g \text{ mapping, shunting}
\end{aligned}$$

□

The Boolean lattice structure of multirelations is therefore preserved in the pointwise Boolean lattice structure of predicate transformers. As regards composition we have the following result for the restricted set of up-closed multirelations [18, Lemma 6.4]. Their (multirelational) composition amounts to the backward composition of the corresponding isotone predicate transformers.

Theorem 11. *Let $f : 2^B \to 2^A$ and $g : 2^C \to 2^B$ be predicate transformers such that the multirelation $\Phi(f)$ is up-closed. Then $f \circ g = gf$.*

Proof. We calculate as follows:

$$\begin{aligned}
f \circ g &= \Psi(\Phi(f) \,;\Phi(g)) && \\
&= \Psi(\Phi(f) \,; \mathsf{E}g^{\mathsf{c}}) && \\
&= \Psi(\Phi(f)(\mathsf{E} \setminus \mathsf{E}g^{\mathsf{c}})) && \text{Theorem 5 (1)} \\
&= \Psi(\Phi(f)(\mathsf{E} \setminus \mathsf{E})g^{\mathsf{c}}) && g \text{ mapping, Lemma 1} \\
&= \mathsf{syq}(\Phi(f)\mathsf{S}g^{\mathsf{c}}, \mathsf{E}) && \\
&= g\,\mathsf{syq}(\Phi(f)\mathsf{S}, \mathsf{E}) && g \text{ mapping, Lemma 1} \\
&= g\,\mathsf{syq}(\Phi(f), \mathsf{E}) && \Phi(f) \text{ up-closed, Theorem 5} \\
&= g\,\mathsf{syq}(\mathsf{E}f^{\mathsf{c}}, \mathsf{E}) && \\
&= gf\,\mathsf{syq}(\mathsf{E}, \mathsf{E}) && f \text{ mapping, Lemma 1} \\
&= gf && \text{axiom of membership relation}
\end{aligned}$$

□

4.3 Alternative Composition of Multirelations

Observe that predicate transformers are mappings, and therefore – when considered as functions in the usual mathematical sense – their composition is associative. The composition of up-closed multirelations is also associative, but the composition of multirelations is not associative in general; see Theorem 6. Similarly, the identity mapping is neutral for composition, but the membership relation is right-neutral only for up-closed multirelations. Hence, the above one-to-one correspondence between predicate transformers and multirelations is not a monoid isomorphism; this is why we restricted attention to up-closed multirelations in Theorem 11.

On the other hand, the one-to-one correspondence induces a monoid isomorphism with a different composition, defined for all multirelations $Q : A \leftrightarrow 2^B$ and $R : B \leftrightarrow 2^C$ by $(Q : R) := \Phi(\Psi(R)\,\Psi(Q)) : A \leftrightarrow 2^C$, which elaborates to

$$\begin{aligned}
(Q : R) &= \Phi(\Psi(R)\,\Psi(Q)) && \\
&= \Phi(\mathsf{syq}(R, \mathsf{E})\,\mathsf{syq}(Q, \mathsf{E})) && \\
&= \mathsf{E}\,\mathsf{syq}(Q, \mathsf{E})^{\mathsf{c}} \mathsf{syq}(R, \mathsf{E})^{\mathsf{c}} && \\
&= \mathsf{E}\,\mathsf{syq}(\mathsf{E}, Q)\,\mathsf{syq}(\mathsf{E}, R) && \text{property of symmetric quotients [30]} \\
&= Q\,\mathsf{syq}(\mathsf{E}, R) && \text{property of symmetric quotients [30]}
\end{aligned}$$

The logical meaning of the new multirelational composition $(Q : R)$ is that for all $x \in A$ and $Z \in 2^C$ we have $(Q : R)_{x,Z}$ if and only if $Q_{x,R^{\mathsf{c}}(Z)}$, where $R^{\mathsf{c}}(Z)$ denotes the set of elements from the set B that R relates with the set Z.

Yet another composition is introduced in [23]. Like the one of Section 3.1 it is not associative for general multirelations as shown in [11], but satisfies weaker algebraic properties.

R is ...	if and only if	R is ...	if and only if
up-closed	$R;\mathsf{E} = R$		
total	$R;\mathsf{T} = \mathsf{T}$	co-total	$R;\mathsf{O} = \mathsf{O}$
$\cup$-distributive	$R;(P \cup Q) = R;P \cup R;Q$		
$\cap$-distributive	$R;(P \cap Q) = R;P \cap R;Q$		
reflexive	$\mathsf{E} \subseteq R$	co-reflexive	$R \subseteq \mathsf{E}$
transitive	$R;R \subseteq R$	dense	$R \subseteq R;R$
idempotent	$R;R = R$		
a contact	$R;R \cup \mathsf{E} = R$	a kernel	$R;R \cap \mathsf{E} = R;\mathsf{E}$
a test	$R;\mathsf{T} \cap \mathsf{E} = R$	a co-test	$R;\mathsf{O} \cup \mathsf{E} = R$
a vector	$R;\mathsf{T} = R$		

Fig. 1. Properties of multirelations

4.4 Translating Properties

Theorem 11 is an example of a result which only holds for a restricted set of multirelations. The property of being up-closed is frequently used, but not the only one that is useful. This property and other properties have been discussed in [3,18,19,22,25,26] and – in the context of topology – already in [1].

In the companion paper [5] we have started a systematic investigation of several multirelational properties by giving relational and algebraic definitions, showing how the properties are related and under which operations they are closed. We now discuss how the properties translate to predicate transformers. Specifically, we look at the properties given in Figure 1 above, where the two distributivity properties universally quantify over the multirelations P and Q. We proceed from top to bottom.

Up-closed Multirelations. We start with the well-known connection of up-closed multirelations and isotone predicate transformers; see [3,14,18,22,25]. The following result expresses it in terms of relation algebras. Note that being up-closed is precisely axiom (K_2) of contact relations [1].

Theorem 12. *Let $f : 2^B \to 2^A$ be a predicate transformer. Then $\Phi(f)$ is up-closed if and only if $\mathsf{S}_B f \subseteq f \mathsf{S}_A$.*

Proof. We calculate as follows:

$$\begin{array}{rcll} \Phi(f) \text{ up-closed} & \iff & \Phi(f)\mathsf{S}_B \subseteq \Phi(f) & \text{Theorem 5} \\ & \iff & \mathsf{E}f^{\mathsf{c}}\mathsf{S}_B \subseteq \mathsf{E}f^{\mathsf{c}} & \\ & \iff & \mathsf{E}f^{\mathsf{c}}\mathsf{S}_B f \subseteq \mathsf{E} & f \text{ mapping, shunting} \\ & \iff & f^{\mathsf{c}}\mathsf{S}_B f \subseteq \mathsf{E} \setminus \mathsf{E} & \text{Galois right residual} \\ & \iff & \mathsf{S}_B f \subseteq f\mathsf{S}_A & f \text{ mapping, shunting} \quad \square \end{array}$$

The inclusion $\mathsf{S}_B f \subseteq f\mathsf{S}_A$ of Theorem 12 expresses relation-algebraically that $X \subseteq Y$ implies $f(X) \subseteq f(Y)$ for all $X, Y \in 2^B$ [21,27].

Total and Co-total Multirelations. The multirelation $R : A \leftrightarrow 2^B$ is called co-total if the empty set is not in its image, that is, $\neg R_{x,\emptyset}$, for all $x \in A$.

This is precisely axiom (K_0) of [1]. The empty multirelation O is not a right-zero of composition, but it is so for co-total multirelations (see [26], where such multirelations are called “total”). As shown in [5], being co-total amounts to $R\,;\mathsf{O} = \mathsf{O}$ and this is equivalent to $R \subseteq \mathsf{TE}$.

Neither is the universal multirelation T a right-zero of composition, but it is so for total multirelations (see [25], where such multirelations are called “proper”). This is because $R\,;\mathsf{T} = \mathsf{T}$ is equivalent to $R\mathsf{T} = \mathsf{T}$ as shown in [5]. How these properties translate to isotone predicate transformers is shown in [25, Theorem 10]. The following result generalises this to arbitrary predicate transformers. It follows that the restriction to co-total multirelations corresponds to Dijkstra’s Law of the Excluded Miracle [9].

Theorem 13. *Let $f : 2^B \to 2^A$ be a predicate transformer. Then $\Phi(f)$ is co-total if and only if $f(\emptyset) = \emptyset$ and $\Phi(f)$ is total if and only if $f(B) = A$.*

Proof. The first claim follows by

$$
\begin{aligned}
\Phi(f)\,;\mathsf{O} = \mathsf{O} &\iff \Phi(f) \subseteq \mathsf{TE} && \text{shown in [5]}\\
&\iff \mathsf{E}f^{\mathsf{c}} \subseteq \mathsf{TE}\\
&\iff \forall x \in A : \forall Y \in 2^B : (\mathsf{E}f^{\mathsf{c}})_{x,Y} \Rightarrow (\mathsf{TE})_{x,Y}\\
&\iff \forall x \in A : \forall Y \in 2^B : x \in f(Y) \Rightarrow Y \neq \emptyset\\
&\iff \forall Y \in 2^B : (\exists x \in A : x \in f(Y)) \Rightarrow Y \neq \emptyset\\
&\iff \forall Y \in 2^B : f(Y) \neq \emptyset \Rightarrow Y \neq \emptyset\\
&\iff \forall Y \in 2^B : Y = \emptyset \Rightarrow f(Y) = \emptyset\\
&\iff f(\emptyset) = \emptyset.
\end{aligned}
$$

To prove the second claim, we start with $\Phi(f)\,;\mathsf{T} = \Phi(f)\mathsf{T}$. Hence, $\Phi(f)\,;\mathsf{T} = \mathsf{T}$ is equivalent to $\mathsf{T} \subseteq \mathsf{E}f^{\mathsf{c}}\mathsf{T}$. That $\mathsf{T} \subseteq \mathsf{E}f^{\mathsf{c}}\mathsf{T}$ is equivalent to $f(B) = A$ can again be verified using pointwise reasoning; we omit the details. ◻

Distributive Multirelations. Next, we investigate $\cup$-distributivity. Assume that $f : 2^B \to 2^A$ is a predicate transformer. Consider for each element $x \in A$ the subset $I_x := \{Y \in 2^B \mid \forall Z \in 2^Y : x \notin f(Z)\}$ of 2^B. The following result relates the sets I_x to $\cup$-distributivity of the multirelation corresponding to f. Recall that a subset I of 2^B is an *ideal* in the lattice $(2^B, \cup, \cap)$ if it is a lower set and closed under binary unions and that a subset F of 2^B is a *filter* in this lattice if it is an upper set and closed under binary intersections.

Theorem 14. *Let $f : 2^B \to 2^A$ be a predicate transformer. Then $\Phi(f)$ is $\cup$-distributive if and only if for all $x \in A$ the set I_x is an ideal in $(2^B, \cup, \cap)$.*

Proof. The multirelation $\Phi(f)$ is $\cup$-distributive if and only if for all predicate transformers $g, h : 2^C \to 2^B$ we have

$$\Phi(f)\,;(\Phi(g) \cup \Phi(h)) = \Phi(f)\,;\Phi(g) \cup \Phi(f)\,;\Phi(h). \tag{1}$$

In the first part of the proof we transform (1) as follows:

$$
\begin{aligned}
&\Phi(f)\,;(\Phi(g) \cup \Phi(h)) = \Phi(f)\,;\Phi(g) \cup \Phi(f)\,;\Phi(h)\\
\iff\ &\Phi(f)\,;(\Phi(g) \cup \Phi(h)) \subseteq \Phi(f)\,;\Phi(g) \cup \Phi(f)\,;\Phi(h) && \text{Theorem 6 (8)}\\
\iff\ &\mathsf{E}f^{\mathsf{c}}(\mathsf{E} \setminus \mathsf{E}(g \cup h)^{\mathsf{c}}) \subseteq \mathsf{E}f^{\mathsf{c}}(\mathsf{E} \setminus \mathsf{E}g^{\mathsf{c}}) \cup \mathsf{E}f^{\mathsf{c}}(\mathsf{E} \setminus \mathsf{E}h^{\mathsf{c}}) && \text{Theorem 5 (1)}\\
\iff\ &\mathsf{E}f^{\mathsf{c}}(\mathsf{E} \setminus \mathsf{E}(g \cup h)^{\mathsf{c}}) \subseteq \mathsf{E}f^{\mathsf{c}}\mathsf{S}g^{\mathsf{c}} \cup \mathsf{E}f^{\mathsf{c}}\mathsf{S}h^{\mathsf{c}} && \text{Lemma 1}
\end{aligned}
$$

Furthermore, for all sets $Y \in 2^C$ and $Z \in 2^B$ we obtain

$$\begin{aligned}(\mathsf{E} \setminus \mathsf{E}(g \cup h)^{\mathsf{c}})_{Z,Y} &\iff \forall w \in B : w \in Z \Rightarrow w \in g(Y) \cup h(Y) \\ &\iff Z \subseteq g(Y) \cup h(Y).\end{aligned}$$

Hence, (1) holds if and only if for all $x \in A$ and $Y \in 2^C$ the following holds:

$$\begin{array}{l}(\exists Z \in 2^B : x \in f(Z) \wedge Z \subseteq g(Y) \cup h(Y)) \\ \Rightarrow \\ (\exists Z \in 2^B : x \in f(Z) \wedge Z \subseteq g(Y)) \vee (\exists Z \in 2^B : x \in f(Z) \wedge Z \subseteq h(Y))\end{array} \tag{2}$$

Next, we define for each $x \in A$ the set $M_x := \{Z \in 2^B \mid x \in f(Z)\}$. Substituting $G = g(Y)$ and $H = h(Y)$ in (2) we get the condition

$$(\exists Z \in M_x : Z \subseteq G \cup H) \;\Rightarrow\; (\exists Z \in M_x : Z \subseteq G) \vee (\exists Z \in M_x : Z \subseteq H). \tag{3}$$

This works because for each g and Y in (2), the set G of (3) can be instantiated to $g(Y)$, and for each G in (3), the mapping g of (2) can be instantiated to the constant mapping $\lambda Y.G : 2^C \to 2^B$, and similarly for h and H.

A consequence is that (2) holds for all predicate transformers $g, h : 2^C \to 2^B$, $x \in A$ and $Y \in 2^C$ if and only if (3) holds for all $x \in A$ and sets $G, H \in 2^B$. Summing up, in the first part we have shown that $\Phi(f)$ is $\cup$-distributive if and only if (3) holds for all $x \in A$ and $G, H \in 2^B$.

In the second part of the proof, we define for all $x \in A$ and sets $Y \in 2^B$ the set $M_{x,Y} := M_x \cap \{Z \in 2^B \mid Z \subseteq Y\}$. Then (3) simplifies to

$$M_{x,G\cup H} \neq \emptyset \;\Rightarrow\; M_{x,G} \neq \emptyset \vee M_{x,H} \neq \emptyset$$

which, in turn, is equivalent to

$$M_{x,G} = \emptyset \wedge M_{x,H} = \emptyset \;\Rightarrow\; M_{x,G\cup H} = \emptyset.$$

Observe furthermore that

$$\begin{aligned}M_{x,G} = \emptyset &\iff M_x \cap \{Z \in 2^B \mid Z \subseteq G\} = \emptyset \\ &\iff \forall Z \in 2^G : Z \notin M_x \\ &\iff \forall Z \in 2^G : x \notin f(Z) \\ &\iff G \in I_x\end{aligned}$$

and similarly for $M_{x,H}$ and $M_{x,G\cup H}$. Hence, (3) is equivalent to

$$G \in I_x \wedge H \in I_x \;\Rightarrow\; G \cup H \in I_x. \tag{4}$$

A consequence is that (3) holds for all $x \in A$ and $G, H \in 2^B$ if and only if (4) holds for all $x \in A$ and $G, H \in 2^B$, but the latter states that I_x is closed under binary union, for all $x \in A$. So, as result of this part we have that $\Phi(f)$ is $\cup$-distributive if and only if for all $x \in A$ the set I_x is closed under binary union.

Finally, each I_x is a lower set since $M_{x,Y}$ is $\subseteq$-isotone in its argument Y. Hence, I_x is closed under binary union if and only if it is an ideal. □

For $\cap$-distributive multirelations we obtain the following dual result, where we consider for each $x \in A$ the set $F_x := \{Y \in 2^B \mid \exists Z \in 2^Y : x \in f(Z)\}$. We omit its proof.

Theorem 15. *Let $f : 2^B \to 2^A$ be a predicate transformer. Then $\Phi(f)$ is $\cap$-distributive if and only if for all $x \in A$ the set F_x is a filter in $(2^B, \cup, \cap)$.*

How distributivity over (arbitrary) union/intersection of up-closed multirelations translates to isotone predicate transformers has been investigated in [25]. In [5] we show that multirelations satisfying axiom (K_4) of [1] are $\cup$-distributive and that this is an equivalence for up-closed multirelations.

Reflexive and Co-reflexive Multirelations. Every reflexive multirelation satisfies axiom (K_1) of [1] and this is an equivalence for up-closed multirelations. The following result shows how reflexivity translates to predicate transformers.

Theorem 16. *Let $f : 2^A \to 2^A$ be a predicate transformer. Then*

(1) $\Phi(f)$ is reflexive if and only if $X \subseteq f(X)$ for each $X \in 2^A$,
(2) $\Phi(f)$ is co-reflexive if and only if $X \supseteq f(X)$ for each $X \in 2^A$.

Proof. (1) Reflexivity translates as follows:

$$\begin{aligned} \mathsf{E} \subseteq \Phi(f) &\iff \mathsf{I} \subseteq \mathsf{E} \setminus \mathsf{E} f^{\mathsf{c}} && \text{Galois right residual} \\ &\iff \mathsf{I} \subseteq \mathsf{S} f^{\mathsf{c}} && f \text{ mapping, Lemma 1} \\ &\iff f \subseteq \mathsf{S} && f \text{ mapping, shunting} \\ &\iff \forall X \in 2^A : X \subseteq f(X) \end{aligned}$$

(2) Co-reflexivity translates as follows:

$$\begin{aligned} \Phi(f) \subseteq \mathsf{E} &\iff \mathsf{E} f^{\mathsf{c}} \subseteq \mathsf{E} \\ &\iff f^{\mathsf{c}} \subseteq \mathsf{S} && \text{Galois right residual} \\ &\iff f \subseteq \mathsf{S}^{\mathsf{c}} \\ &\iff \forall X \in 2^A : f(X) \subseteq X \end{aligned}$$

□

Transitive, Dense and Idempotent Multirelations. Transitivity of multirelations is precisely axiom (K_3) of [1]. We first give a result for isotone predicate transformers. The first part shows how transitivity translates to predicate transformers. The remaining parts deal with the properties "dense" and "idempotent".

Theorem 17. *Let $f : 2^A \to 2^A$ be an isotone predicate transformer. Then*

(1) $\Phi(f)$ is transitive if and only if $f(f(X)) \subseteq f(X)$ for each $X \in 2^A$,
(2) $\Phi(f)$ is dense if and only if $f(f(X)) \supseteq f(X)$ for each $X \in 2^A$,
(3) $\Phi(f)$ is idempotent if and only if $f(f(X)) = f(X)$ for each $X \in 2^A$.

Proof. (1) Observe that $\mathsf{E}f^{\mathsf{c}} \subseteq \mathsf{E}f^{\mathsf{c}}\mathsf{S} \subseteq \mathsf{E}\mathsf{S}f^{\mathsf{c}} = \mathsf{E}f^{\mathsf{c}}$ by Lemma 2, since f is isotone and E is up-closed. Hence, we get

$$\Phi(f)\,;\Phi(f) = (\mathsf{E}f^{\mathsf{c}})\,;(\mathsf{E}f^{\mathsf{c}}) = \mathsf{E}f^{\mathsf{c}}(\mathsf{E}\setminus\mathsf{E}f^{\mathsf{c}}) = \mathsf{E}f^{\mathsf{c}}(\mathsf{E}\setminus\mathsf{E})f^{\mathsf{c}} = \mathsf{E}f^{\mathsf{c}}\mathsf{S}f^{\mathsf{c}} = \mathsf{E}f^{\mathsf{c}}f^{\mathsf{c}}$$

by Theorem 5 and Lemma 1, since f is a mapping. This implies the claim by

$$\begin{aligned}
\Phi(f)\,;\Phi(f) \subseteq \Phi(f) &\iff \mathsf{E}f^{\mathsf{c}}f^{\mathsf{c}} \subseteq \mathsf{E}f^{\mathsf{c}} \\
&\iff f^{\mathsf{c}}f^{\mathsf{c}} \subseteq \mathsf{E}\setminus\mathsf{E}f^{\mathsf{c}} && \text{Galois right residual} \\
&\iff f^{\mathsf{c}}f^{\mathsf{c}} \subseteq \mathsf{S}f^{\mathsf{c}} && f \text{ mapping, Lemma 1} \\
&\iff ff \subseteq f\mathsf{S}^{\mathsf{c}} \\
&\iff \forall X,Y \in 2^A : (ff)_{X,Y} \Rightarrow (f\mathsf{S}^{\mathsf{c}})_{X,Y} \\
&\iff \forall X,Y \in 2^A : Y = f(f(X)) \Rightarrow Y \subseteq f(X) \\
&\iff \forall X \in 2^A : f(f(X)) \subseteq f(X).
\end{aligned}$$

(2) We use again the equation $\Phi(f)\,;\Phi(f) = \mathsf{E}f^{\mathsf{c}}f^{\mathsf{c}}$ from the proof of (1):

$$\begin{aligned}
\Phi(f) \subseteq \Phi(f)\,;\Phi(f) &\iff \mathsf{E}f^{\mathsf{c}} \subseteq \mathsf{E}f^{\mathsf{c}}f^{\mathsf{c}} \\
&\iff f^{\mathsf{c}} \subseteq \mathsf{E}\setminus\mathsf{E}f^{\mathsf{c}}f^{\mathsf{c}} && \text{Galois right residual} \\
&\iff f^{\mathsf{c}} \subseteq \mathsf{S}f^{\mathsf{c}}f^{\mathsf{c}} && f \text{ mapping, Lemma 1} \\
&\iff f \subseteq ff\mathsf{S}^{\mathsf{c}} \\
&\iff \forall X \in 2^A : f(X) \subseteq f(f(X))
\end{aligned}$$

(3) This follows immediately by combining (1) and (2). ◻

If f is not required to be isotone, the conditions become more complex; for example, the multirelation $\Phi(f)$ is transitive if and only if $Y \subseteq f(X)$ implies $f(Y) \subseteq f(X)$, for each $X, Y \in 2^A$. We omit further details about this and instead give an example using dense multirelations.

Example 18. We consider a basic program transformation that imports an invariant into the body of a loop. Assume that multirelations P and R represent the invariant and the body, respectively. We wish to prove

$$P\,;(\nu X.R\,;X) = P\,;(\nu X.(P\,;R)\,;X), \qquad (\star)$$

which uses the $\subseteq$-greatest fixpoint ν to represent the semantics of an endless loop. Such a transformation might be followed by simplifications of the loop body using the invariant P.

The invariant P is frequently represented as a test, which acts as a filter in sequential compositions [16, Lemma 2.3.2]. But transformation $(\star)$ can be proved with much weaker assumptions, namely

(1) P is dense, that is, $P \subseteq P\,;P$,
(2) P is preserved by R, that is, $P\,;R \subseteq R\,;P$, and
(3) $P\,;(Y\,;Z) \subseteq (P\,;Y)\,;Z$ for all multirelations Y and Z.

The latter is required because multirelational composition only satisfies the semi-associativity law given in Theorem 6 (7), namely (4) $(X\,;Y)\,;Z \subseteq X\,;(Y\,;Z)$ for all X, Y, and Z. To show that $(\star)$ follows from assumptions (1) to (3), we start by

$$\begin{aligned} P;(\nu X.R;X) &\subseteq (P;R);(\nu X.R;X) && \text{unfold } \nu \text{ and (3)} \\ &\subseteq (P;(P;R));(\nu X.R;X) && \text{(1) and (4)} \\ &\subseteq (P;R);(P;(\nu X.R;X)) && \text{(2) and (3) and (4).} \end{aligned}$$

The greatest fixpoint property of ν implies (5) $P\,;(\nu X.R;X) \subseteq \nu X.(P;R)\,;X$. Then one inclusion of $(\star)$ is shown by

$$\begin{aligned} P;(\nu X.R;X) &\subseteq (P;P);(\nu X.R;X) && \text{(1)} \\ &\subseteq P;(\nu X.(P;R);X) && \text{(4) and (5).} \end{aligned}$$

For the other inclusion observe that

$$\begin{aligned} P;(\nu X.(P;R);X) &\subseteq (P;(P;R));(\nu X.(P;R);X) && \text{unfold } \nu \text{ and (3)} \\ &\subseteq ((P;R);P);(\nu X.(P;R);X) && \text{(2) and (3)} \\ &\subseteq (P;R);(P;(\nu X.(P;R);X)) && \text{(4).} \end{aligned}$$

The greatest fixpoint property implies (6) $P\,;(\nu X.(P;R);X) \subseteq \nu X.(P;R)\,;X$. Hence, we get

$$\begin{aligned} \nu X.(P;R);X &\subseteq (R;P);(\nu X.(P;R);X) && \text{unfold } \nu \text{ and (2)} \\ &\subseteq R;(\nu X.(P;R);X) && \text{(4) and (6).} \end{aligned}$$

Finally, the greatest fixpoint property implies $\nu X.(P;R)\,;X \subseteq \nu X.R\,;X$, from which the remaining inclusion of $(\star)$ follows by composing P on the left. ◻

Reasoning of the kind shown in this example can be done algebraically (abstracting from concrete multirelations) and is well supported by automated theorem provers; see [5,11], for example.

Contacts and Kernels. In [5] we show algebraically that a multirelation is a contact if and only if it is reflexive, transitive and up-closed (hence also idempotent). As a consequence, the corresponding isotone predicate transformer $f : 2^A \to 2^A$ satisfies $X \subseteq f(X) = f(f(X))$, for all $X \in 2^A$. In other words, contact multirelations (in the sense of [1]) correspond to predicate transformers that are closure operations – as already shown in [1].

Dually, a multirelation is a kernel if and only if it is co-reflexive, dense and up-closed (hence also idempotent). This implies that the corresponding isotone predicate transformer $f : 2^A \to 2^A$ satisfies $X \supseteq f(X) = f(f(X))$ for all $X \in 2^A$; that is, it is a kernel operator in the sense of lattice theory.

Tests and Co-tests. The following result shows how tests translate from multirelations to predicate transformers.

Theorem 19. *Let $f : 2^A \to 2^A$ be a predicate transformer. Then $\Phi(f)$ is a test if and only if $f(X) = X \cap f(A)$ for each $X \in 2^A$.*

Proof. In terms of relation algebras, we have

$$\begin{aligned}\Phi(f) \text{ is a test} &\iff \Phi(f) = \Phi(f)\mathsf{T} \cap \mathsf{E} \\ &\iff \mathsf{E}f^{\mathsf{c}} = \mathsf{E}f^{\mathsf{c}}\mathsf{T} \cap \mathsf{E} \\ &\iff \mathsf{E}f^{\mathsf{c}} \subseteq \mathsf{E} \ \wedge\ \mathsf{E}f^{\mathsf{c}}\mathsf{T} \cap \mathsf{E} \subseteq \mathsf{E}f^{\mathsf{c}}.\end{aligned}$$

The first expression amounts to co-reflexivity; see Theorem 16 (2):

$$\mathsf{E}f^{\mathsf{c}} \subseteq \mathsf{E} \iff \forall X \in 2^A : f(X) \subseteq X$$

The second expression can be transformed as follows:

$$\begin{aligned}&\mathsf{E}f^{\mathsf{c}}\mathsf{T} \cap \mathsf{E} \subseteq \mathsf{E}f^{\mathsf{c}} \\ \iff &\forall x \in A : \forall X \in 2^A : (\mathsf{E}f^{\mathsf{c}}\mathsf{T} \cap \mathsf{E})_{x,X} \Rightarrow (\mathsf{E}f^{\mathsf{c}})_{x,X} \\ \iff &\forall x \in A : \forall X \in 2^A : (\mathsf{E}f^{\mathsf{c}}\mathsf{T})_{x,X} \wedge \mathsf{E}_{x,X} \Rightarrow x \in f(X) \\ \iff &\forall x \in A : \forall X \in 2^A : (\exists Y, Z \in 2^A : \mathsf{E}_{x,Y} \wedge {f^{\mathsf{c}}}_{Y,Z}) \wedge x \in X \Rightarrow x \in f(X) \\ \iff &\forall x \in A : \forall X \in 2^A : (\exists Z \in 2^A : x \in f(Z)) \wedge x \in X \Rightarrow x \in f(X) \\ \iff &\forall X, Z \in 2^A : \forall x \in A : x \in f(Z) \wedge x \in X \Rightarrow x \in f(X) \\ \iff &\forall X, Z \in 2^A : f(Z) \cap X \subseteq f(X)\end{aligned}$$

Both expressions together imply that f is isotone, since from $X \subseteq Y$ we get

$$f(X) = f(X) \cap X \subseteq f(X) \cap Y \subseteq f(Y),$$

for all $X, Y \in 2^A$. Hence, we have $f(X) \subseteq f(A)$ for all $X \in 2^A$ and, thus,

$$\begin{aligned}\Phi(f) \text{ is a test} &\iff f \text{ is isotone } \wedge\ \forall X, Z \in 2^A : f(Z) \cap X \subseteq f(X) \subseteq X \\ &\iff f \text{ is isotone } \wedge\ \forall X \in 2^A : f(A) \cap X \subseteq f(X) \subseteq X \\ &\iff f \text{ is isotone } \wedge\ \forall X \in 2^A : f(A) \cap X = f(X) \\ &\iff \forall X \in 2^A : f(A) \cap X = f(X)\end{aligned}$$

since the latter also implies that f is isotone. □

For co-tests we obtain the following dual result; we omit its proof.

Theorem 20. *Let $f : 2^A \to 2^A$ be a predicate transformer. Then $\Phi(f)$ is a co-test if and only if $f(X) = X \cup f(\emptyset)$ for each $X \in 2^A$.*

Vectors. A multirelation R is a vector if and only if $R\ ;\ \mathsf{T} = R$, which is equivalent to $R\mathsf{T} = R$. The final result in our series of translations shows how the vector property translates to predicate transformers.

Theorem 21. *Let $f : 2^A \to 2^A$ be a predicate transformer. Then $\Phi(f)$ is a vector if and only if f is a constant function.*

Proof. The claim is proved by the following calculation:

$$
\begin{array}{rlll}
\Phi(f) \text{ is a vector} & \iff & \Phi(f)\mathsf{T} = \Phi(f) & \\
& \iff & \Phi(f)\mathsf{T} \subseteq \Phi(f) & \\
& \iff & \mathsf{E}f^{\mathsf{c}}\mathsf{T} \subseteq \mathsf{E}f^{\mathsf{c}} & \\
& \iff & \mathsf{E}f^{\mathsf{c}}\mathsf{T}f \subseteq \mathsf{E} & f \text{ mapping, shunting} \\
& \iff & f^{\mathsf{c}}\mathsf{T}f \subseteq \mathsf{S} & \text{Galois right residual} \\
& \iff & (\mathsf{T}f)^{\mathsf{c}}(\mathsf{T}f) \subseteq \mathsf{S} & \mathsf{TT} = \mathsf{T} \\
& \iff & (\mathsf{T}f)^{\mathsf{c}}(\mathsf{T}f) \subseteq \mathsf{S} \cap \mathsf{S}^{\mathsf{c}} & \text{LHS is symmetric} \\
& \iff & (\mathsf{T}f)^{\mathsf{c}}(\mathsf{T}f) \subseteq \mathsf{I} & \text{Lemma 2} \\
& \iff & \mathsf{T}f \text{ univalent} & \\
& \iff & f \text{ is constant} & f \text{ mapping}
\end{array}
$$
□

5 Conclusion

In the present paper we have investigated how properties from research on multirelations and contact relations translate to predicate transformers. Similar to the correspondence between contact relations and closure operations in [29] we have expressed the correspondence of multirelations and predicate transformers within the language of relation algebras. This approach allowed us to generalise many results from up-closed to arbitrary multirelations. Looking beyond up-closed multirelations is a necessary step to cover approaches to program semantics such as concurrent dynamic logic [23]. We thus hope to establish new links between program semantics and other areas (for example, topology and games) which use multirelations as a conceptual and methodological basis.

Further exploration of the new composition operator for multirelations and a comparison with the other composition operators mentioned in Section 4.3 should be carried out in the future. Another topic for further work are probabilistic multirelations, which are relations of the type $A \leftrightarrow D(A)$ where $D(A)$ is the set of all probabilistic sub-distributions over A; see [32] for details.

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