Multirelations with infinite computations

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Abstract

Multirelations model computations with both angelic and demonic non-determinism. We extend multirelations to represent finite and infinite computations independently. We derive an approximation order for multirelations assuming only that the endless loop is its least element and that the lattice operations are isotone. We use relations, relation algebra and RelView for representing and calculating with multirelations and for finding the approximation order.

Keywords: approximation order, median, multirelations, program semantics, relations, RelView, sequential computations

1. Introduction

Computation models that support both angelic and demonic choice are useful for modelling contracts between agents, interaction, games and protocols [1, 20]. Finite computations in such models can be represented by multirelations [22, 23, 24]. However, multirelations have no means for representing computations that fail to terminate independently of finite computations; this is similar to relational computation models such as those of [17]. The first goal of this paper is to extend multirelations to represent finite and infinite computations independently.

The basic idea to achieve this is the same as for relations, namely to extend the state space. The main challenge is to provide a suitable approximation order, which is needed to define the semantics of recursion. In some relational models, the Egli-Milner order is used for approximation. But already in the relational setting it is not clear which approximation order to use when more precise models are considered; these require variations of the Egli-Milner order [11, 14]. It is even less obvious how to generalise the Egli-Milner order to multirelations. Thus the second goal of this paper is to find means of defining approximation which are independent of particular computation models.

We apply relational methods to achieve the two goals. First, we express the basic definitions, operations and properties of multirelations in terms of relations. This involves transforming logical expressions to relational formulas and algebraic calculations with these formulas. Second, we derive relational programs for investigating the approximation order in RelView [3]. From particular instances of approximation we distill a relational definition that is suitable for multirelations; its properties are again shown by applying algebra.

The main contributions of this paper are as follows:

- We introduce strict multirelations in Section 5.2 as a reaction to the failure of having an approximation order for all up-closed multirelations.
- We give an approximation order for strict, up-closed multirelations, which is expressed by Theorem 15 in terms of relations. It turns out to be the same as an order in a previous study of pointed distributive lattices [10]. We thus obtain many useful properties including representations of least fixpoints.
- In Theorem 18 we show that our approximation order is the $\subseteq$-least partial order which has the endless loop as least element and isotone lattice operations. This is a new characterisation with very weak assumptions expected to hold in many computation models.

- We give definitions, operations and properties of multirelations, multirelations with a special state, the endless loop, up-closed multirelations and strict multirelations in terms of relations. We implement programs using such multirelations in RelView.

Section 2 recalls heterogeneous relations. Section 3 expresses multirelations in terms of relations and shows how they represent computations. Section 4 extends the state space to represent infinite computations. Section 5 applies RelView to derive an approximation order, shows that it is suitable for multirelations, expresses it in terms of relations, characterises it based on weak assumptions, and instantiates previous results for the representation of fixpoints.

Isotone predicate transformers are isomorphic to up-closed multirelations [22, 1, 23, 16]. Many basic properties of multirelations shown in Section 3 are known from existing research on these models; here we give a relational development. Sections 4 and 5 extend these models by representing finite and infinite computations independently. For example, a non-deterministic choice between the endless loop and the program that does not change the state will be different from both of these programs. Existing approaches that make this distinction do not support both angelic and demonic choice [18, 21, 15, 8].

2. Relations

In this section we recall basic definitions, operations and properties of heterogeneous relations [28, 27, 26]; see also [4, 25, 2].

2.1. Basic operations

Given sets A and B, a relation R of type $A \leftrightarrow B$ is a subset of the Cartesian product $A \times B$; we write $R : A \leftrightarrow B$ and abbreviate $(x, y) \in R$ by R_{xy}. The relations of type $A \leftrightarrow B$ form a complete Boolean algebra with union $\cup$, arbitrary union $\bigcup$, intersection $\cap$, arbitrary intersection $\bigcap$, complement $\overline{}$ and partial order $\subseteq$ bounded by the empty relation $\mathsf{O} = \emptyset$ and the universal relation $\mathsf{T} = A \times B$. The composition of two relations $Q : A \leftrightarrow B$ and $R : B \leftrightarrow C$ is the relation $QR : A \leftrightarrow C$ defined by $(QR)_{xz} \Leftrightarrow \exists y \in B : Q_{xy} \wedge R_{yz}$. The identity relation $\mathsf{I} : A \leftrightarrow A$ is defined by $\mathsf{I}_{xy} \Leftrightarrow x = y$. The converse of a relation $R : A \leftrightarrow B$ is the relation $R^\smile : B \leftrightarrow A$ defined by $R^\smile{}_{xy} \Leftrightarrow R_{yx}$. Union, intersection and composition are defined only if the types of the involved relations match as indicated above; we tacitly assume this in the remainder of this paper. Composition has higher precedence than union and intersection.

Union, intersection, composition and converse are $\subseteq$-isotone; complement is $\subseteq$-antitone. We furthermore use the following properties:

- Composition distributes over $\bigcup$ and converse distributes over $\bigcup$, $\bigcap$ and $\overline{}$.
- $(QR)^\smile = R^\smile Q^\smile$ and $R^{\smile\smile} = R$ and $\mathsf{O}^\smile = \mathsf{O}$ and $\mathsf{I}^\smile = \mathsf{I}$ and $\mathsf{T}^\smile = \mathsf{T}$.
- $\mathsf{O}R = \mathsf{O}$ and $R\mathsf{O} = \mathsf{O}$ and $\mathsf{I}R = R = R\mathsf{I}$ and $\mathsf{TT} = \mathsf{T}$.
- $PQ \subseteq R \Leftrightarrow P^\smile \overline{R} \subseteq \overline{Q} \Leftrightarrow \overline{R} Q^\smile \subseteq \overline{P}$; these are the Schröder equivalences.
- $\mathsf{T}R\mathsf{T} = \mathsf{T}$ if $R \neq \mathsf{O}$; this is the Tarski rule.
- $R = R^\smile$ if $R \subseteq \mathsf{I}$.

A relation R is total if $R\mathsf{T} = \mathsf{T}$, univalent if $R^\smile R \subseteq \mathsf{I}$, surjective if $\mathsf{T}R = \mathsf{T}$, injective if $RR^\smile \subseteq \mathsf{I}$, a mapping if R is total and univalent, bijective if R is injective and surjective, a vector if $R\mathsf{T} = R$, reflexive if $\mathsf{I} \subseteq R$, transitive if $RR \subseteq R$, antisymmetric if $R \cap R^\smile \subseteq \mathsf{I}$, a preorder if R is reflexive and transitive, and a partial order if R is an antisymmetric preorder. A vector $R : A \leftrightarrow B$ represents a set which contains those elements of A that are related by R to every element of B. We use the following properties:

- $(\bigcap_{i \in I} Q_i)R = \bigcap_{i \in I}(Q_i R)$ if R is injective and $\overline{QR} = \overline{Q}R$ if R is bijective.
- $PQ \subseteq R \Leftrightarrow Q \subseteq P^\smile R$ and $QP^\smile \subseteq R \Leftrightarrow Q \subseteq RP$ if P is bijective.

- $(P\mathsf{T} \cap Q)R = P\mathsf{T} \cap QR$ and $Q(R \cap \mathsf{T}P) = QR \cap \mathsf{T}P$ and $(Q \cap \mathsf{T}P^{\smile})R = Q(P\mathsf{T} \cap R)$.
- Vectors are closed under $\bigcup$, $\bigcap$ and $^{-}$.

2.2. Residuals and symmetric quotient

The right residual is $Q\backslash R = \overline{Q^{\smile}\overline{R}}$, so $(Q\backslash R)_{xy} \Leftrightarrow (\forall z : Q_{zx} \Rightarrow R_{zy})$. The left residual is $Q/R = \overline{\overline{Q}R^{\smile}}$, so $(Q/R)_{xy} \Leftrightarrow (\forall z : R_{yz} \Rightarrow Q_{xz})$. The symmetric quotient is $(Q{\div}R) = (Q\backslash R) \cap (Q^{\smile}/R^{\smile})$, whence $(Q{\div}R)_{xy} \Leftrightarrow (\forall z : Q_{zx} \Leftrightarrow R_{zy})$. We use the following properties of these constructions:

(1) $\backslash$ is $\subseteq$-antitone in its first argument and $\subseteq$-isotone in its second argument.
(2) $/$ is $\subseteq$-isotone in its first argument and $\subseteq$-antitone in its second argument.
(3) $Q(Q\backslash R) \subseteq R$.
(4) $QP \subseteq R \Leftrightarrow P \subseteq Q\backslash R$.
(5) $PQ \subseteq R \Leftrightarrow P \subseteq R/Q$.
(6) $Q\backslash(P \cap R) = (Q\backslash P) \cap (Q\backslash R)$.
(7) $\mathsf{I}\backslash R = R$.
(8) $R/\mathsf{I} = R$.
(9) $R\backslash\mathsf{T} = \mathsf{T}$.
(10) $(Q\backslash R)P = Q\backslash(RP)$ if P is bijective.
(11) $(Q/R)P = Q/(P^{\smile}R)$ if P is bijective.
(12) $(Q{\div}R)^{\smile} = (R{\div}Q)$.
(13) $(\overline{R}{\div}\overline{Q}) = (R{\div}Q)$.
(14) $(P{\div}Q)(Q{\div}R) = (P{\div}R) \cap \mathsf{T}(Q{\div}R)$.
(15) $(Q{\div}\mathsf{O}) = Q\backslash\mathsf{O}$.

2.3. Power sets

The power set of A is $2^A = \{B \mid B \subseteq A\}$. The membership relation $\mathsf{E} : A \leftrightarrow 2^A$ is defined by $\mathsf{E}_{xY} \Leftrightarrow x \in Y$. The subset relation $\mathsf{S} : 2^A \leftrightarrow 2^A$ is $\mathsf{S} = \mathsf{E}\backslash\mathsf{E}$, whence $\mathsf{S}_{XY} \Leftrightarrow X \subseteq Y$. The complement relation $\mathsf{C} : 2^A \leftrightarrow 2^A$ is $\mathsf{C} = (\mathsf{E}{\div}\overline{\mathsf{E}})$, whence $\mathsf{C}_{XY} \Leftrightarrow X = \overline{Y}$. We use the following properties from the literature:

(16) $(R{\div}\mathsf{E})$ is a mapping.
(17) $(\mathsf{E}{\div}R)$ is bijective.
(18) $\mathsf{E}(\mathsf{E}{\div}R) = R$.
(19) S is a partial order.
(20) $\mathsf{ES} = \mathsf{E}$.
(21) C is a bijective mapping.
(22) $\mathsf{CC} = \mathsf{I}$.
(23) $\mathsf{C}^{\smile} = \mathsf{C}$.
(24) $\mathsf{EC} = \overline{\mathsf{E}}$.

To these, we add the following properties.

Theorem 1.

1. E *is total.*
2. $\mathsf{E}/\mathsf{E} = \mathsf{I}$.
3. $(Q{\div}\mathsf{E})(\mathsf{E}{\div}R) = (Q{\div}R)$.
4. $\mathsf{S}(\mathsf{E}{\div}R) = \mathsf{E}\backslash R$.
5. $\mathsf{E}(\mathsf{E}\backslash R) = R$.
6. $\mathsf{CS} = \mathsf{S}^{\smile}\mathsf{C}$.

PROOF.

1. $\mathsf{T} = (\mathsf{T} \div \mathsf{E})\mathsf{T} \subseteq (\mathsf{T} \backslash \mathsf{E})\mathsf{T} \subseteq (\mathsf{I} \backslash \mathsf{E})\mathsf{T} = \mathsf{E}\mathsf{T}$ using (16), (1) and (7).
2. $\mathsf{I} \subseteq \mathsf{E}/\mathsf{E} = \mathsf{E}/(\mathsf{I} \backslash \mathsf{E}) \subseteq \mathsf{E}/(\mathsf{I} \div \mathsf{E}) = \mathsf{E}/(\mathsf{E} \div \mathsf{I})^{\smile} = (\mathsf{E}/\mathsf{I})(\mathsf{E} \div \mathsf{I}) = \mathsf{E}(\mathsf{E} \div \mathsf{I}) = \mathsf{I}$ using (5), (7), (2), (12), (11), (17), (8) and (18).
3. $(Q \div \mathsf{E})(\mathsf{E} \div R) = (Q \div R) \cap \mathsf{T}(\mathsf{E} \div R) = (Q \div R)$ using (14) and (17).
4. $\mathsf{S}(\mathsf{E} \div R) = (\mathsf{E} \backslash \mathsf{E})(\mathsf{E} \div R) = \mathsf{E} \backslash (\mathsf{E}(\mathsf{E} \div R)) = \mathsf{E} \backslash R$ using (10), (17) and (18).
5. $\mathsf{E}(\mathsf{E} \backslash R) = \mathsf{E}\mathsf{S}(\mathsf{E} \div R) = \mathsf{E}(\mathsf{E} \div R) = R$ using the preceding claim, (20) and (18).
6. By a Schröder equivalence, (20) implies $\overline{\mathsf{E}}\mathsf{S}^{\smile} \subseteq \overline{\mathsf{E}}$. Thus $\mathsf{E}\mathsf{C}\mathsf{S}^{\smile}\mathsf{C} = \overline{\mathsf{E}}\mathsf{S}^{\smile}\mathsf{C} \subseteq \overline{\mathsf{E}}\mathsf{C} = \mathsf{E}\mathsf{C}\mathsf{C} = \mathsf{E}\mathsf{I} = \mathsf{E}$ using (24) and (22). Hence $\mathsf{C}\mathsf{S}^{\smile}\mathsf{C} \subseteq \mathsf{E} \backslash \mathsf{E} = \mathsf{S}$ using (4) and therefore $\mathsf{S}^{\smile}\mathsf{C} \subseteq \mathsf{C}^{\smile}\mathsf{S} = \mathsf{C}\mathsf{S}$ using (21) and (23). Thus also $\mathsf{C}\mathsf{S} = \mathsf{C}^{\smile}\mathsf{S} = (\mathsf{S}^{\smile}\mathsf{C})^{\smile} \subseteq (\mathsf{C}\mathsf{S})^{\smile} = \mathsf{S}^{\smile}\mathsf{C}^{\smile} = \mathsf{S}^{\smile}\mathsf{C}$ using (23). □

2.4. Cartesian product

The projections $p_1 : A \times B \leftrightarrow A$ and $p_2 : A \times B \leftrightarrow B$ are defined by $p_{1(x,y)z} \Leftrightarrow x = z$ and $p_{2(x,y)z} \Leftrightarrow y = z$. The tupling of relations $Q : A \leftrightarrow B$ and $R : A \leftrightarrow C$ is $[Q, R] : A \leftrightarrow B \times C$ defined by $[Q, R] = Qp_1{}^{\smile} \cap Rp_2{}^{\smile}$ using projections p_1, p_2 of appropriate type; hence $[Q, R]_{x(y,z)} \Leftrightarrow Q_{xy} \wedge R_{xz}$. The parallel product of relations $Q : A \leftrightarrow B$ and $R : C \leftrightarrow D$ is $Q \| R : A \times C \leftrightarrow B \times D$ defined by $Q \| R = p_1 Q r_1{}^{\smile} \cap p_2 R r_2{}^{\smile}$ using projections p_1, p_2 and r_1, r_2 of appropriate types; hence $(Q \| R)_{(w,x)(y,z)} \Leftrightarrow Q_{wy} \wedge R_{xz}$.

3. Multirelations

In this section we recall basic definitions, operations and properties of multirelations and express them in terms of relations.

A *multirelation* [22, 23] is a relation of type $A \leftrightarrow 2^B$. It maps an element of A to a set of subsets of B. Union, intersection and complement apply to multirelations as to relations. Particular multirelations are O, T and E.

3.1. Composition

Multirelational composition of $Q : A \leftrightarrow 2^B$ and $R : B \leftrightarrow 2^C$ is the multirelation $Q \,;\, R : A \leftrightarrow 2^C$ given by

$$(Q \,;\, R)_{xZ} \Leftrightarrow \exists Y \subseteq B : Q_{xY} \wedge \forall y \in Y : R_{yZ}$$

The following result expresses multirelational composition in terms of relations using a right residual; see also [26].

Theorem 2. $Q \,;\, R = Q(\mathsf{E} \backslash R)$.

PROOF.

$$\begin{array}{ll} & (Q \,;\, R)_{xZ} \\ \Leftrightarrow & \exists Y \subseteq B : Q_{xY} \wedge \forall y \in Y : R_{yZ} \\ \Leftrightarrow & \exists Y \subseteq B : Q_{xY} \wedge \forall y \in B : \mathsf{E}_{yY} \Rightarrow R_{yZ} \\ \Leftrightarrow & \exists Y \subseteq B : Q_{xY} \wedge (\mathsf{E} \backslash R)_{YZ} \\ \Leftrightarrow & (Q(\mathsf{E} \backslash R))_{xZ} \end{array}$$

□

The following result gives properties of multirelational composition.

Theorem 3.

1. *The operation* ; *is* $\subseteq$*-isotone.*
2. $Q \,;\, \mathsf{T} = Q\mathsf{T}$.
3. $Q \,;\, R = Q$ *if* Q *is a vector.*

4. $\mathsf{O} \mathbin{;} R = \mathsf{O}$.
5. $\mathsf{E} \mathbin{;} R = R$.
6. $\mathsf{T} \mathbin{;} R = \mathsf{T}$.
7. $(\bigcup_{i\in I} Q_i) \mathbin{;} R = \bigcup_{i\in I}(Q_i \mathbin{;} R)$.

PROOF.

1. This follows from $Q \mathbin{;} R = Q(\mathsf{E}\backslash R)$ using (1) and that relational composition is $\subseteq$-isotone.
2. $Q \mathbin{;} \mathsf{T} = Q(\mathsf{E}\backslash\mathsf{T}) = Q\mathsf{T}$ using (9).
3. $Q(\mathsf{E}\div\mathsf{O})^{\smile} \subseteq Q\mathsf{T} = Q$ since Q is a vector. Hence

$$Q \subseteq Q(\mathsf{E}\div\mathsf{O}) = Q(\mathsf{E}\backslash\mathsf{O}) = Q \mathbin{;} \mathsf{O} \subseteq Q \mathbin{;} R \subseteq Q \mathbin{;} \mathsf{T} = Q\mathsf{T} = Q$$

   using (17), (15), the first claim and the second claim.
4. This follows from the preceding claim since O is a vector.
5. $\mathsf{E} \mathbin{;} R = \mathsf{E}(\mathsf{E}\backslash R) = R$ using Theorem 1.5.
6. This follows from the third claim since T is a vector.
7. $(\bigcup_{i\in I} Q_i) \mathbin{;} R = (\bigcup_{i\in I} Q_i)(\mathsf{E}\backslash R) = \bigcup_{i\in I}(Q_i(\mathsf{E}\backslash R)) = \bigcup_{i\in I}(Q_i \mathbin{;} R)$. □

3.2. Dual

The *dual* of a multirelation $R : A \leftrightarrow 2^B$ is the multirelation $R^d : A \leftrightarrow 2^B$ given by

$$R^d{}_{xY} \Leftrightarrow \neg R_{x\overline{Y}}$$

where $\overline{Y}$ is the complement of Y relative to B. The following result expresses the dual in terms of relations using the complement relation C.

Theorem 4. $R^d = \overline{R}\mathsf{C} = \overline{R\mathsf{C}}$.

PROOF.

$$\begin{array}{ll} & R^d{}_{xY} \\ \Leftrightarrow & \neg R_{x\overline{Y}} \\ \Leftrightarrow & \exists Z \subseteq B : Z = \overline{Y} \wedge \neg R_{xZ} \\ \Leftrightarrow & \exists Z \subseteq B : \mathsf{C}_{ZY} \wedge \overline{R}_{xZ} \\ \Leftrightarrow & (\overline{R}\mathsf{C})_{xY} \end{array}$$

Finally, $\overline{R}\mathsf{C} = \overline{R\mathsf{C}}$ using (21). □

The following result gives properties of dual multirelations.

Theorem 5.

1. *The operation $\cdot^d$ is $\subseteq$-antitone.*
2. $R^d = \overline{R}$ *if R is a vector.*
3. $\mathsf{O}^d = \mathsf{T}$.
4. $\mathsf{E}^d = \mathsf{E}$.
5. $\mathsf{T}^d = \mathsf{O}$.
6. $R^{d^d} = R$.
7. $(\bigcup_{i\in I} R_i)^d = \bigcap_{i\in I} R_i{}^d$.
8. $(\bigcap_{i\in I} R_i)^d = \bigcup_{i\in I} R_i{}^d$.

PROOF.

1. This follows from $R^d = \overline{R}\mathsf{C}$ since relational composition is $\subseteq$-isotone and complement is $\subseteq$-antitone.

2. $R^d = \overline{R\mathsf{C}} = \overline{R\mathsf{TC}} = \overline{R\mathsf{T}} = \overline{R}$ using that R is a vector and (21).
3. This follows from the preceding claim since O is a vector.
4. $\mathsf{E}^d = \overline{\mathsf{EC}} = \overline{\overline{\mathsf{E}}} = \mathsf{E}$ using (24).
5. This follows from the third claim since T is a vector.
6. $R^{d^d} = \overline{R^d}\mathsf{C} = R\mathsf{CC} = R\mathsf{I} = R$ using (22).
7. $(\bigcup_{i\in I} R_i)^d = \overline{(\bigcup_{i\in I} R_i)\mathsf{C}} = \overline{\bigcup_{i\in I}(R_i\mathsf{C})} = \bigcap_{i\in I} \overline{R_i\mathsf{C}} = \bigcap_{i\in I} {R_i}^d$.
8. $(\bigcap_{i\in I} R_i)^d = (\bigcap_{i\in I} {R_i}^{d^d})^d = (\bigcup_{i\in I} {R_i}^d)^{d^d} = \bigcup_{i\in I} {R_i}^d$ using the preceding two claims. □

3.3. Up-closed multirelations

A multirelation $R : A \leftrightarrow 2^B$ is *up-closed* if

$$R_{xY} \wedge Y \subseteq Z \Rightarrow R_{xZ}$$

for each $x \in A$ and $Y, Z \subseteq B$. This means that if an element of A is related to a set Y, it must be related to all supersets of Y. The following result expresses this property in terms of relations using the subset relation S; see also [26].

Theorem 6. *R is up-closed if and only if $R = R\mathsf{S}$.*

Proof.

$$\begin{array}{ll} & R : A \leftrightarrow 2^B \text{ is up-closed} \\ \Leftrightarrow & \forall x \in A : \forall Y, Z \subseteq B : R_{xY} \wedge Y \subseteq Z \Rightarrow R_{xZ} \\ \Leftrightarrow & \forall x \in A : \forall Y, Z \subseteq B : R_{xY} \wedge \mathsf{S}_{YZ} \Rightarrow R_{xZ} \\ \Leftrightarrow & \forall x \in A : \forall Z \subseteq B : (\exists Y \subseteq B : R_{xY} \wedge \mathsf{S}_{YZ}) \Rightarrow R_{xZ} \\ \Leftrightarrow & \forall x \in A : \forall Z \subseteq B : (R\mathsf{S})_{xZ} \Rightarrow R_{xZ} \\ \Leftrightarrow & R\mathsf{S} \subseteq R \end{array}$$

This is equivalent to $R\mathsf{S} = R$ since $R = R\mathsf{I} \subseteq R\mathsf{S}$ using (19). □

The following result shows that the membership relation is an identity for the composition of up-closed multirelations. It also shows that multirelational operations preserve the property of being up-closed.

Theorem 7.

1. *$R \mathbin{;} \mathsf{E} = R$ if and only if R is up-closed.*
2. *$P \mathbin{;} (Q \mathbin{;} R) = (P \mathbin{;} Q) \mathbin{;} R$ if Q is up-closed.*
3. *$(Q \mathbin{;} R)^d = Q^d \mathbin{;} R^d$ if Q is up-closed.*
4. *Every vector is up-closed.*
5. *O, E and T are up-closed.*
6. *$Q \mathbin{;} R$ and PR and R^d are up-closed if R is up-closed.*
7. *$\bigcup_{i\in I} R_i$ and $\bigcap_{i\in I} R_i$ are up-closed if R_i is up-closed for each $i \in I$.*
8. *$(\bigcap_{i\in I} Q_i) \mathbin{;} R = \bigcap_{i\in I}(Q_i \mathbin{;} R)$ if Q_i is up-closed for each $i \in I$.*

Proof.

1. $R \mathbin{;} \mathsf{E} = R(\mathsf{E}\backslash\mathsf{E}) = R\mathsf{S} = R$ if R is up-closed. From $R \mathbin{;} \mathsf{E} = R$ we obtain $R\mathsf{S} = R(\mathsf{E}\backslash\mathsf{E}) = R \mathbin{;} \mathsf{E} = R$.
2. $\mathsf{E}(\mathsf{E}\backslash Q)(\mathsf{E}\backslash R) \subseteq Q(\mathsf{E}\backslash R)$ using (3), whence $(\mathsf{E}\backslash Q)(\mathsf{E}\backslash R) \subseteq \mathsf{E}\backslash(Q(\mathsf{E}\backslash R))$ using (4). Moreover

$$\mathsf{E}\backslash(Q(\mathsf{E}\backslash R)) = \mathsf{E}\backslash(Q\mathsf{S}(\mathsf{E}\div R)) = \mathsf{E}\backslash(Q(\mathsf{E}\div R)) = (\mathsf{E}\backslash Q)(\mathsf{E}\div R) \subseteq (\mathsf{E}\backslash Q)(\mathsf{E}\backslash R)$$

using Theorem 1.4, that Q is up-closed, (10) and (17). Thus $\mathsf{E}\backslash(Q(\mathsf{E}\backslash R)) = (\mathsf{E}\backslash Q)(\mathsf{E}\backslash R)$, whence

$$P \mathbin{;} (Q \mathbin{;} R) = P \mathbin{;} (Q(\mathsf{E}\backslash R)) = P(\mathsf{E}\backslash(Q(\mathsf{E}\backslash R))) = P(\mathsf{E}\backslash Q)(\mathsf{E}\backslash R) = (P(\mathsf{E}\backslash Q)) \mathbin{;} R = (P \mathbin{;} Q) \mathbin{;} R$$

3. $Q\mathsf{S} = Q$ implies $\overline{Q}\mathsf{S}^{\smile} \subseteq \overline{Q} = \overline{Q}\mathsf{I} = \overline{Q}\mathsf{I}^{\smile} \subseteq \overline{Q}\mathsf{S}^{\smile}$ using a Schröder equivalence and (19), whence $\overline{Q}\mathsf{S}^{\smile} = \overline{Q}$. Therefore

$$\begin{aligned}(Q\ ;\ R)^d &= \overline{Q\ ;\ R}\mathsf{C} = \overline{Q(\mathsf{E}\backslash R)}\mathsf{C} = \overline{Q\mathsf{S}(\mathsf{E}\div R)}\mathsf{C} = \overline{Q(\mathsf{E}\div R)}\mathsf{C} = \overline{Q}(\mathsf{E}\div R)\mathsf{C} = \overline{Q}(\overline{\mathsf{E}}\div \overline{R})\mathsf{C} \\ &= \overline{Q}(\overline{\mathsf{E}}\div \mathsf{E})(\mathsf{E}\div \overline{R})\mathsf{C} = \overline{Q}\mathsf{C}^{\smile}(\mathsf{E}\div \overline{R})\mathsf{C} = \overline{Q}\mathsf{C}(\mathsf{E}\div \overline{R})\mathsf{C} = \overline{Q}\mathsf{S}^{\smile}\mathsf{C}(\mathsf{E}\div \overline{R})\mathsf{C} = \overline{Q}\mathsf{C}\mathsf{S}(\mathsf{E}\div \overline{R})\mathsf{C} \\ &= \overline{Q}\mathsf{C}(\mathsf{E}\backslash \overline{R})\mathsf{C} = Q^d(\mathsf{E}\backslash \overline{R})\mathsf{C} = Q^d(\mathsf{E}\backslash(\overline{R}\mathsf{C})) = Q^d(\mathsf{E}\backslash R^d) = Q^d\ ;\ R^d\end{aligned}$$

using Theorem 1.4, that Q is up-closed, (17), (13), Theorem 1.3, (12), (23), Theorems 1.6 and 1.4, (10) and (21).
4. Let R be a vector. Then $R = R\mathsf{I} \subseteq R\mathsf{S} \subseteq R\mathsf{T} = R$ using (19), whence $R = R\mathsf{S}$.
5. This follows from the first claim and Theorems 3.4, 3.5 and 3.6.
6. $(Q\ ;\ R)\ ;\ \mathsf{E} = Q\ ;\ (R\ ;\ \mathsf{E}) = Q\ ;\ R$ using the first two claims since R is up-closed. Clearly $PR\mathsf{S} = PR$ if R is up-closed. Moreover $R^d\ ;\ \mathsf{E} = R^d\ ;\ \mathsf{E}^d = (R\ ;\ \mathsf{E})^d = R^d$ using Theorem 5.4, the third claim and the first claim since R is up-closed.
7. $(\bigcup_{i\in I} R_i)\mathsf{S} = \bigcup_{i\in I}(R_i\mathsf{S}) = \bigcup_{i\in I} R_i$ using that R_i is up-closed for each $i \in I$. Moreover $(\bigcap_{i\in I} R_i)\mathsf{S} \subseteq \bigcap_{i\in I}(R_i\mathsf{S}) = \bigcap_{i\in I} R_i = (\bigcap_{i\in I} R_i)\mathsf{I} \subseteq (\bigcap_{i\in I} R_i)\mathsf{S}$ using that R_i is up-closed for each $i \in I$ and (19).
8. The preceding claim implies

$$\begin{aligned}(\textstyle\bigcap_{i\in I} Q_i)\ ;\ R &= (\textstyle\bigcap_{i\in I} Q_i)(\mathsf{E}\backslash R) = (\textstyle\bigcap_{i\in I} Q_i)\mathsf{S}(\mathsf{E}\div R) = (\textstyle\bigcap_{i\in I} Q_i)(\mathsf{E}\div R) \\ &= \textstyle\bigcap_{i\in I}(Q_i(\mathsf{E}\div R)) = \textstyle\bigcap_{i\in I}(Q_i\mathsf{S}(\mathsf{E}\div R)) = \textstyle\bigcap_{i\in I}(Q_i(\mathsf{E}\backslash R)) = \textstyle\bigcap_{i\in I}(Q_i\ ;\ R)\end{aligned}$$

using Theorem 1.4 and (17). □

3.4. Computations

In the remainder of this paper we look at multirelations of type $A \leftrightarrow 2^B$ where $A = B$. Such multirelations model computations that involve two kinds of non-determinism associated with players in a game [22] and sometimes called angelic and demonic [1, 24]. The underlying intuition is that there are two players, the 'angel' who makes a choice and the 'demon' who subsequently makes a choice based on the angel's selection. The following description is taken from [13].

Consider a multirelation $R : A \leftrightarrow 2^A$, a state $x \in A$ and the set of subsets $Ys = \{Y \subseteq A \mid R_{xY}\}$ to which x is related. The outer set structure of Ys represents angelic choice: the angel chooses a set $Y \in Ys$. The inner set structure of Ys represents demonic choice: the demon subsequently chooses an element $y \in Y$ which is the next state.

For example, let $A = \{0, 1, 2, 3, 4\}$ and let $R : A \leftrightarrow 2^A$ be given by

$$\begin{array}{rcl} 0 & \mapsto & \{\{1,2\},\{1,3,4\}\} \\ 1 & \mapsto & \{\{1\},\{2\}\} \\ 2 & \mapsto & \{\{1,2\}\} \\ 3 & \mapsto & \{\emptyset\} \\ 4 & \mapsto & \emptyset \end{array}$$

It describes the following computation. In state 0 an angelic choice between two sets $\{1, 2\}$ and $\{1, 3, 4\}$ is made. If the angel chooses $\{1, 3, 4\}$ the demon chooses which of 1, 3 and 4 is the next state. If the angel chooses $\{1, 2\}$ the demon chooses one of 1 and 2 as the next state. State 1 has a purely angelic choice between states 1 and 2, because each inner set is a singleton set in which the demon's choice is fixed. State 2 has a purely demonic choice between states 1 and 2, because the outer set is a singleton set in which the angel's choice is fixed. In state 3 the computation fails to progress since the demon cannot choose from the empty set. In the game interpretation this means that the angel wins; in terms of refinement this means that any specification is satisfied. In state 4 the computation fails to progress since the angel cannot choose from the empty set. In the game interpretation this means that the demon wins; in terms of refinement this means that no specification is satisfied.

The multirelation R is not up-closed, but can be extended to an up-closed multirelation $Q = R\mathsf{S}$ by adding the required supersets. Adding a superset Z of a set Y to which x is related does not change the computational interpretation. This just increases the angelic choice by options which are not interesting for the angel because they subsequently allow more choices for the demon. For example, in Q the state 0 is related to $\{1,2,3\}$, but angelic choice prefers $\{1,2\}$ so as to restrict demonic choice as much as possible.

In the remainder of this paper we focus on up-closed multirelations. They feature a nice interplay between the outer and the inner set structures. Forming the union of up-closed multirelations simultaneously increases angelic choice and decreases demonic choice, while intersection simultaneously decreases angelic choice and increases demonic choice. Union and intersection thus provide angelic and demonic choice, respectively, at the level of computations.

Angelic choice and demonic choice are duals of each other. In an alternative computational interpretation the outer set structure describes demonic choice and the inner set structure describes angelic choice [7]. Then union represents demonic choice and intersection represents angelic choice.

4. Infinite computations

Multirelations have no particular means for representing infinite computations. The situation is akin to relational computation models like those of [17]. A special multirelation such as O or T could be interpreted as an infinite computation. However, the properties $\mathsf{O} \cap R = \mathsf{O}$ and $\mathsf{O} \cup R = R = \mathsf{T} \cap R$ and $\mathsf{T} \cup R = \mathsf{T}$ would imply that finite and infinite computations starting in the same state cannot be represented independently.

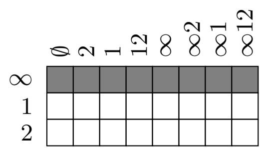
In this section, we extend the multirelational model to represent infinite computations independently of finite computations. The general idea is the same as for relations, namely to extend the state space by a special element ∞ that represents the outcome of an infinite computation. We therefore consider up-closed multirelations of type $A \leftrightarrow 2^A$ where $\infty \in A$. Note that we do not add traces – finite or infinite – but maintain that multirelations ignore the intermediate states of a computation.

4.1. A special state

State spaces extended this way are typically structured by a partial order, for example, the flat order with ∞ as least element. We use an alternative approach to distinguish ∞ by considering the special multirelation $\mathsf{N} : A \leftrightarrow 2^A$ given by

$$\mathsf{N}_{xY} \Leftrightarrow x = \infty$$

For $A = \{\infty, 1, 2\}$ its matrix is



It is easy to show that N is a point, that is, N is a non-zero, injective vector:

- $\mathsf{N} \neq \mathsf{O}$,
- $\mathsf{NT} = \mathsf{N}$,
- $\mathsf{NN}^{\smile} \subseteq \mathsf{I}$.

In fact, we will only use these properties of N, which therefore can be taken as axioms for N instead of the pointwise definition above. The following result derives further properties of N.

Theorem 8.

1. N *is up-closed.*
2. N *is bijective.*
3. $\mathsf{N}^{\smile}\mathsf{N} = \mathsf{T}$.
4. $\mathsf{TN}^{\smile}\mathsf{T} = \mathsf{T}$.

5. $N \cap T(N \cap R) = N \cap R$.
6. $N \,;\, R = N$.

Proof.

1. This follows from Theorem 7.4 since N is a vector.
2. N is surjective as $TN = TNT = T$ using that N is a vector and the Tarski rule with $N \neq O$.
3. Using the preceding claim, $T \subseteq N^\smile N$ is equivalent to $NT \subseteq N$, which holds since N is a vector.
4. $TN^\smile T = T$ using the Tarski rule since $N \neq O$ implies $N^\smile \neq O$.
5. $N \cap R \subseteq N$ and $N \cap R = I(N \cap R) \subseteq T(N \cap R)$. Moreover $N \cap T(N \cap R) \subseteq N$ and

$$N \cap T(N \cap R) = N \cap T(NT \cap R) = N \cap TN^\smile R = NT \cap TN^\smile R = NTN^\smile R = NN^\smile R \subseteq IR = R$$

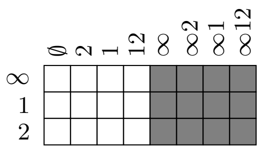
   using that N is a vector and injective.
6. This follows from Theorem 3.3 since N is a vector. □

4.2. The endless loop

Based on N we define the multirelation $L : A \leftrightarrow 2^A$ which represents the endless loop. It relates each element of A to every subset of A that contains ∞ and is given by

$$L = T(N \cap E)$$

A simple calculation shows $L_{xY} \Leftrightarrow \infty \in Y$. For $A = \{\infty, 1, 2\}$ its matrix is



The following result gives properties of L.

Theorem 9.

1. $L = TN^\smile E$.
2. $L^\smile T = E^\smile N$.
3. L *is total.*
4. L *is up-closed.*
5. $L^\smile$ *is a vector.*
6. $TL \subseteq E \backslash L$.
7. $L(E \backslash N) = T$.
8. $N = \overline{(\overline{E} \cap L)T}$.
9. $N \cap E = N \cap L$.
10. $L^d = L$.
11. $L \,;\, R = TN^\smile R = T(N \cap R)$.
12. $L \subseteq L \,;\, R$ *if and only if* $N \cap E \subseteq R$.
13. $L = L \,;\, R$ *if and only if* $N \cap E = N \cap R$.
14. $L = L \,;\, L = L \,;\, E$.

Proof.

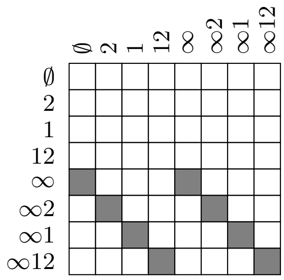
1. $L = T(N \cap E) = T(NT \cap E) = TN^\smile E$ using that N is a vector.
2. $L^\smile T = E^\smile NTT = E^\smile NT = E^\smile N$ using the preceding claim and that N is a vector.

3. $\mathsf{LT} = \mathsf{TN}^\smile\mathsf{ET} = \mathsf{TN}^\smile\mathsf{T} = \mathsf{T}$ using the first claim and Theorems 1.1 and 8.4.
4. L is up-closed using the first claim and Theorems 7.5 and 7.6.
5. $\mathsf{TL} = \mathsf{TT}(\mathsf{N} \cap \mathsf{E}) = \mathsf{T}(\mathsf{N} \cap \mathsf{E}) = \mathsf{L}$, whence $\mathsf{L}^\smile$ is a vector.
6. $\mathsf{ETL} = \mathsf{TL} = \mathsf{L}$ using Theorem 1.1 and the preceding claim. Hence $\mathsf{TL} \subseteq \mathsf{E}\backslash\mathsf{L}$ using (4).
7. $\mathsf{L}(\mathsf{E}\backslash\mathsf{N}) = \mathsf{TN}^\smile\mathsf{E}(\mathsf{E}\backslash\mathsf{N}) = \mathsf{TN}^\smile\mathsf{N} = \mathsf{TT} = \mathsf{T}$ using the first claim and Theorems 1.5 and 8.3.
8. $\overline{(\overline{\mathsf{E}} \cap \mathsf{L})\mathsf{T}} = \overline{(\overline{\mathsf{E}} \cap \mathsf{TN}^\smile\mathsf{E})\mathsf{T}} = \overline{\overline{\mathsf{E}}\mathsf{E}^\smile\mathsf{NT}} = \overline{\overline{\mathsf{E}}\mathsf{E}^\smile\mathsf{N}} = \overline{\overline{\mathsf{E}}\mathsf{E}^\smile}\mathsf{N} = (\mathsf{E}/\mathsf{E})\mathsf{N} = \mathsf{IN} = \mathsf{N}$ using the first claim, that N is a vector and Theorems 8.2 and 1.2.
9. $\mathsf{N} \cap \mathsf{E} = \mathsf{N} \cap \mathsf{T}(\mathsf{N} \cap \mathsf{E}) = \mathsf{N} \cap \mathsf{L}$ using Theorem 8.5.
10. $\overline{\mathsf{L}^d} = \mathsf{LC} = \mathsf{TN}^\smile\mathsf{EC} = \mathsf{TN}^\smile\overline{\mathsf{E}} = \mathsf{T}(\mathsf{NT} \cap \overline{\mathsf{E}}) = \mathsf{T}(\mathsf{N} \cap \overline{\mathsf{E}})$ using the first claim, (24) and that N is a vector. Hence
$$\mathsf{L} \cup \overline{\mathsf{L}^d} = \mathsf{TN}^\smile\mathsf{E} \cup \mathsf{TN}^\smile\overline{\mathsf{E}} = \mathsf{TN}^\smile(\mathsf{E} \cup \overline{\mathsf{E}}) = \mathsf{TN}^\smile\mathsf{T} = \mathsf{T}$$
using the first claim and Theorem 8.4. Moreover
$$\mathsf{L} \cap \overline{\mathsf{L}^d} = \mathsf{TL} \cap \mathsf{T}(\mathsf{N} \cap \overline{\mathsf{E}}) = \mathsf{T}(\mathsf{N} \cap \overline{\mathsf{E}} \cap \mathsf{TL}) = \mathsf{T}(\overline{\mathsf{E}} \cap \mathsf{N} \cap \mathsf{L}) = \mathsf{T}(\overline{\mathsf{E}} \cap \mathsf{N} \cap \mathsf{E}) = \mathsf{TO} = \mathsf{O}$$
using the fifth claim and the ninth claim. Thus $\mathsf{L}^d = \mathsf{L}$.
11. $\mathsf{L}\ ;R = \mathsf{L}(\mathsf{E}\backslash R) = \mathsf{TN}^\smile\mathsf{E}(\mathsf{E}\backslash R) = \mathsf{TN}^\smile R = \mathsf{T}(\mathsf{NT} \cap R) = \mathsf{T}(\mathsf{N} \cap R)$ using the first claim, Theorem 1.5 and that N is a vector.
12. Using the preceding claim the backward implication holds by
$$\mathsf{L} = \mathsf{T}(\mathsf{N} \cap \mathsf{E}) \subseteq \mathsf{T}(\mathsf{N} \cap R) = \mathsf{L}\ ;R$$
The forward implication holds since the ninth claim and the eleventh claim and Theorem 8.5 imply
$$\mathsf{N} \cap \mathsf{E} = \mathsf{N} \cap \mathsf{L} \subseteq \mathsf{N} \cap (\mathsf{L}\ ;R) = \mathsf{N} \cap \mathsf{T}(\mathsf{N} \cap R) = \mathsf{N} \cap R$$
13. This holds by replacing $\subseteq$ with $=$ in the preceding two calculations.
14. This follows from the preceding claim and the ninth claim. □

The computational interpretation of L is as follows. In each state, the best choice for the angel is the set $\{\infty\}$, whence the demon will choose ∞; thus ∞ is the unique outcome. For every multirelation R such that $\mathsf{L} \subseteq R$, the angel can force this outcome by choosing the set $\{\infty\}$. For every multirelation R such that $R \subseteq \mathsf{L}$, the demon can force this outcome since any set that can be chosen by the angel contains ∞.

4.3. Adding the special state

In Section 5.3 we will use the relation $\mathsf{K} : 2^A \leftrightarrow 2^A$ whose converse adds ∞ to a set. It is given by
$$\mathsf{K} = (\mathsf{E}{\div}(\mathsf{E} \cup \mathsf{N}))$$
A calculation shows $\mathsf{K}_{XY} \Leftrightarrow X = Y \cup \{\infty\}$. For $A = \{\infty, 1, 2\}$ its matrix is



The following result gives properties of K.

Theorem 10.

1. K *is bijective.*
2. $K \subseteq L^{\smile}T$.
3. $K^{\smile} \subseteq S$ *and* $I \subseteq SK$ *and* K *is antisymmetric.*
4. $L^{\smile}T \cap I \subseteq K$ *and* $L^{\smile}T \cap I \subseteq K^{\smile}$.
5. $KK = K$ *and* $KK^{\smile} \subseteq K \subseteq K^{\smile}K$ *and* $KK^{\smile} \subseteq K^{\smile} \subseteq K^{\smile}K$.
6. $RK^{\smile} = R \cap L$ *if* R *is up-closed.*

Proof.

1. This follows immediately from (17).
2. This is implied by the following calculation, which uses Theorems 8.2 and 9.2:

$$\begin{aligned} K &= (E \div (E \cup N)) \subseteq E^{\smile}/(E \cup N)^{\smile} = \overline{\overline{E}^{\smile}(E \cup N)} = \overline{\overline{E}^{\smile}E \cup \overline{E}^{\smile}N} = \overline{\overline{E}^{\smile}E} \cap \overline{\overline{E}^{\smile}N} = \overline{E^{\smile}\overline{E}}^{\smile} \cap \overline{\overline{E}}^{\smile}N \\ &= (E \backslash E)^{\smile} \cap E^{\smile}N = S^{\smile} \cap L^{\smile}T \end{aligned}$$

3. The preceding calculation also implies $K^{\smile} \subseteq S$, whence $I \subseteq SK$ using the first claim and $K \cap K^{\smile} \subseteq S^{\smile} \cap S \subseteq I$ using (19).
4. The above calculation also yields $L^{\smile}T \cap S^{\smile} = E^{\smile}/(E \cup N)^{\smile}$. Moreover $I = I^{\smile} \subseteq S^{\smile}$ using (19), and $E \subseteq E \cup N$ implies $I \subseteq E\backslash(E \cup N)$ using (4). Hence

$$L^{\smile}T \cap I \subseteq L^{\smile}T \cap S^{\smile} \cap (E\backslash(E \cup N)) = (E^{\smile}/(E \cup N)^{\smile}) \cap (E\backslash(E \cup N)) = (E \div (E \cup N)) = K$$

   Thus $L^{\smile}T \cap I = (L^{\smile}T \cap I)^{\smile} \subseteq K^{\smile}$.
5. $(E \cup N)K = EK \cup NK = E(E \div (E \cup N)) \cup NK = E \cup N \cup NK \subseteq E \cup NT = E \cup N$ using (18) and that N is a vector. Hence $K \subseteq (E \cup N)\backslash(E \cup N)$ using (4). But also

$$K = (E \div (E \cup N)) \subseteq E^{\smile}/(E \cup N)^{\smile} \subseteq (E \cup N)^{\smile}/(E \cup N)^{\smile}$$

   using (2). Together $K \subseteq ((E \cup N) \div (E \cup N))$. Hence

$$K^{\smile} \subseteq ((E \cup N) \div (E \cup N)) = ((E \cup N) \div E)(E \div (E \cup N)) = K^{\smile}K$$

   using (12) and Theorem 1.3. This implies $K \subseteq K^{\smile}K$ and $K^{\smile}K^{\smile} \subseteq K^{\smile}$ using the first claim. Therefore $KK^{\smile} \subseteq K^{\smile}$ again using the first claim and also $KK \subseteq K$. Thus $KK^{\smile} \subseteq K$, which implies $K \subseteq KK$ using the first claim again.
6. $RK^{\smile} \subseteq RS = R$ using the third claim and that R is up-closed. Moreover $RK^{\smile} \subseteq RTL \subseteq TL = L$ using the second claim and Theorem 9.5. Also $R \cap L = (R \cap TL)I = R(L^{\smile}T \cap I) \subseteq RK^{\smile}$ using Theorem 9.5 and the fourth claim. □

5. Approximation

The solution of the recursive specification $R = f(R)$ is the least fixpoint of the function f in a suitable approximation order. Instantiating f with the identity function yields the endless loop, which is represented by the multirelation L. Hence L should be the least element of the approximation order. Because L is neither O nor T, the subset and the superset order cannot be used for approximation. In this section, we use RelView to derive a suitable approximation order.

5.1. Deriving an approximation order

The general idea is to look for a relation that satisfies a few properties expected of approximation, but to not impose unnecessary constraints. Let $\sqsubseteq$ denote the approximation relation on up-closed multirelations; reasonable properties are:

- $\sqsubseteq$ is a partial order,
- L is the $\sqsubseteq$-least element,
- $\cup$, $\cap$, $;$ and $\cdot^d$ are $\sqsubseteq$-isotone.

We implement a RelView program that constructs the $\subseteq$-least relation $\sqsubseteq$ that satisfies these properties. In the following we describe the relational constructions; the implementation is given in the appendix.

Because $\sqsubseteq$ applies only to up-closed multirelations, we first need to enumerate these. The following result gives the set of up-closed multirelations as a vector using the parallel product of relations.

Theorem 11. *The vector* $\overline{(\mathsf{E}^\smile(\mathsf{I}\|\mathsf{S}) \cap \overline{\mathsf{E}}^\smile)\mathsf{T}} : 2^{A\times 2^B} \leftrightarrow \{\star\}$ *represents the set of up-closed multirelations of type* $A \leftrightarrow 2^B$.

Proof. Let $p_1 : A \times 2^B \leftrightarrow A$ and $p_2 : A \times 2^B \leftrightarrow 2^B$ be projections. Then

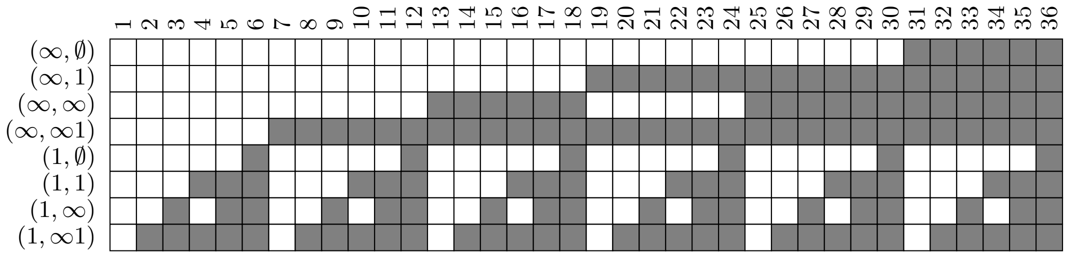
$$\begin{array}{ll}
 & R : A \leftrightarrow 2^B \text{ is up-closed} \\
\Leftrightarrow & \forall x, Y, Z : R_{xY} \wedge Y \subseteq Z \Rightarrow R_{xZ} \\
\Leftrightarrow & \forall x, Y, Z : \mathsf{E}_{(x,Y)R} \wedge \mathsf{S}_{YZ} \Rightarrow \mathsf{E}_{(x,Z)R} \\
\Leftrightarrow & \forall x, Y, Z : \mathsf{E}^\smile{}_{R(x,Y)} \wedge \mathsf{S}_{YZ} \Rightarrow \mathsf{E}^\smile{}_{R(x,Z)} \\
\Leftrightarrow & \forall x, Y, Z : (\exists u : \mathsf{E}^\smile{}_{Ru} \wedge p_{1ux} \wedge p_{2uY}) \wedge \mathsf{S}_{YZ} \Rightarrow \mathsf{E}^\smile{}_{R(x,Z)} \\
\Leftrightarrow & \forall x, Z : \neg\exists Y : (\exists u : \mathsf{E}^\smile{}_{Ru} \wedge p_{1ux} \wedge p_{2uY}) \wedge \mathsf{S}_{YZ} \wedge \neg\mathsf{E}^\smile{}_{R(x,Z)} \\
\Leftrightarrow & \forall x, Z : \neg\exists u : \mathsf{E}^\smile{}_{Ru} \wedge p_{1ux} \wedge (\exists Y : p_{2uY} \wedge \mathsf{S}_{YZ}) \wedge \overline{\mathsf{E}}^\smile{}_{R(x,Z)} \\
\Leftrightarrow & \forall x, Z : \neg\exists u : \mathsf{E}^\smile{}_{Ru} \wedge p_{1ux} \wedge (p_2\mathsf{S})_{uZ} \wedge (\exists v : \overline{\mathsf{E}}^\smile{}_{Rv} \wedge p_{1vx} \wedge p_{2vZ}) \\
\Leftrightarrow & \neg\exists u, v, x, Z : \mathsf{E}^\smile{}_{Ru} \wedge p_{1ux} \wedge (p_2\mathsf{S})_{uZ} \wedge \overline{\mathsf{E}}^\smile{}_{Rv} \wedge p_{1vx} \wedge p_{2vZ} \\
\Leftrightarrow & \neg\exists u, v : \mathsf{E}^\smile{}_{Ru} \wedge \overline{\mathsf{E}}^\smile{}_{Rv} \wedge (\exists x : p_{1ux} \wedge p_1{}^\smile{}_{xv}) \wedge (\exists Z : (p_2\mathsf{S})_{uZ} \wedge p_2{}^\smile{}_{Zv}) \\
\Leftrightarrow & \neg\exists u, v : \mathsf{E}^\smile{}_{Ru} \wedge \overline{\mathsf{E}}^\smile{}_{Rv} \wedge (p_1p_1{}^\smile)_{uv} \wedge (p_2\mathsf{S}p_2{}^\smile)_{uv} \\
\Leftrightarrow & \neg\exists v : (\exists u : \mathsf{E}^\smile{}_{Ru} \wedge (p_1\mathsf{I}p_1{}^\smile \cap p_2\mathsf{S}p_2{}^\smile)_{uv}) \wedge \overline{\mathsf{E}}^\smile{}_{Rv} \\
\Leftrightarrow & \neg\exists v : (\mathsf{E}^\smile(\mathsf{I}\|\mathsf{S}))_{Rv} \wedge \overline{\mathsf{E}}^\smile{}_{Rv} \\
\Leftrightarrow & \neg\exists v : (\mathsf{E}^\smile(\mathsf{I}\|\mathsf{S}) \cap \overline{\mathsf{E}}^\smile)_{Rv} \\
\Leftrightarrow & \neg((\mathsf{E}^\smile(\mathsf{I}\|\mathsf{S}) \cap \overline{\mathsf{E}}^\smile)\mathsf{T})_{R\star} \\
\Leftrightarrow & \overline{(\mathsf{E}^\smile(\mathsf{I}\|\mathsf{S}) \cap \overline{\mathsf{E}}^\smile)\mathsf{T}}_{R\star}
\end{array}$$

The ranges of quantified variables are $x \in A$ and $Y, Z \subseteq B$ and $u, v \in A \times 2^B$. ☐

A construction described in [2] transforms this vector to a relation that contains one column for each up-closed multirelation:

$$\mathsf{E}\,\mathrm{inj}(\overline{(\mathsf{E}^\smile(\mathsf{I}\|\mathsf{S}) \cap \overline{\mathsf{E}}^\smile)\mathsf{T}})^\smile$$

For every vector $V \neq \mathsf{O}$, $\mathrm{inj}(V)$ is an injective mapping such that $\mathrm{inj}(V)^\smile\mathsf{T} = V$. Each column of the resulting relation is the row-wise representation of the entries of an up-closed multirelation. For example, the following relation is obtained for $A = B = \{\infty, 1\}$:



Columns 1, 15, 16, 31, 36 contain the multirelations O, L, E, N, T, respectively.

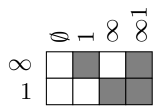
The range of this relation is the set of up-closed multirelations, and hence the domain of the approximation relation. We apply the following algorithm to compute the approximation relation based on the above requirements:

$\sqsubseteq := \{(\mathsf{L}, R) \mid R \text{ up-closed}\}^*$
while $\sqsubseteq$ changes do
  for each $(Q, R) \in \sqsubseteq$ do
    for each up-closed P do
      $\sqsubseteq := \sqsubseteq \cup \{(P \cup Q, P \cup R), (P \cap Q, P \cap R), (Q \mathbin{;} P, R \mathbin{;} P), (P \mathbin{;} Q, P \mathbin{;} R), (Q^d, R^d)\}$
$\sqsubseteq := \sqsubseteq^*$

The initialisation makes L the $\sqsubseteq$-least element. The update in the for-loops ensures that $\cup$, $\cap$, $;$ and $\cdot^d$ are $\sqsubseteq$-isotone. Forming the reflexive-transitive closure * ensures that $\sqsubseteq$ is a preorder. The only requirement not guaranteed is that $\sqsubseteq$ is antisymmetric.

The RelView implementation of this algorithm uses library functions to convert between multirelations of type $A \leftrightarrow 2^B$ and vectors of type $A \times 2^B \leftrightarrow \{\star\}$. The vector representation is used to index a multirelation in the above enumeration with the help of a symmetric quotient. The index is used to extract entries from $\sqsubseteq$ and to add new entries to $\sqsubseteq$. The multirelation representation is used to calculate the multirelations by which $\sqsubseteq$ is updated.

Applying the algorithm for $A = B = \{\infty, 1\}$ gives $\sqsubseteq = \mathsf{T}$, which is not antisymmetric. It is therefore necessary to restrict the set of up-closed multirelations. Comparing with relational computation models such as those of [17, 8] we find that some up-closed multirelations represent non-strict computations. For example, column 21 of the above enumeration contains the multirelation



In state ∞ the outcome is 1; in state 1 the outcome is ∞. Thus the computation delivers an outcome even if the preceding computation fails to terminate.

5.2. Strict multirelations

A multirelation R is *strict* if $\mathsf{L} = \mathsf{L} \mathbin{;} R$. Letting the endless loop be a left annihilator of sequential composition characterises strictness also in various relational computation models [12]. For multirelations, strictness is equivalent to each of $\mathsf{N} \cap \mathsf{E} = \mathsf{N} \cap R$ and $\mathsf{N} \cap \mathsf{L} = \mathsf{N} \cap R$ by Theorems 9.13 and 9.9.

The following result gives the set of strict multirelations as a vector using the tupling of relations.

Theorem 12. *The vector* $((\mathsf{E}^\smile/[\mathsf{I}, \mathsf{E}]) \cap (\overline{\mathsf{E}}^\smile/[\mathsf{I}, \overline{\mathsf{E}}]))\mathsf{NT} : 2^{A \times 2^A} \leftrightarrow \{\star\}$ *represents the set of strict multirelations of type* $A \leftrightarrow 2^A$.

Proof.

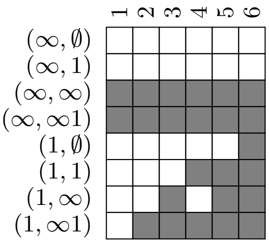
$$
\begin{aligned}
& R : A \leftrightarrow 2^A \text{ is strict} \\
\Leftrightarrow\ & \mathsf{N} \cap \mathsf{E} \subseteq R \wedge \mathsf{N} \cap R \subseteq \mathsf{E} \\
\Leftrightarrow\ & (\forall x, Y : (\mathsf{N} \cap \mathsf{E})_{xY} \Rightarrow R_{xY}) \wedge (\forall x, Y : (\mathsf{N} \cap R)_{xY} \Rightarrow \mathsf{E}_{xY}) \\
\Leftrightarrow\ & (\forall x, Y : \mathsf{N}_{xY} \wedge \mathsf{E}_{xY} \Rightarrow R_{xY}) \wedge (\forall x, Y : \mathsf{N}_{xY} \wedge R_{xY} \Rightarrow \mathsf{E}_{xY}) \\
\Leftrightarrow\ & (\forall x, Y : x = \infty \wedge \mathsf{E}_{xY} \Rightarrow (x, Y) \in R) \wedge (\forall x, Y : x = \infty \wedge (x, Y) \in R \Rightarrow \mathsf{E}_{xY}) \\
\Leftrightarrow\ & (\forall x, Y : x = \infty \wedge \mathsf{E}_{\infty Y} \Rightarrow \mathsf{E}_{(x,Y)R}) \wedge (\forall x, Y : x = \infty \wedge \neg \mathsf{E}_{\infty Y} \Rightarrow (x, Y) \notin R) \\
\Leftrightarrow\ & (\forall x, Y : \mathsf{I}_{\infty x} \wedge \mathsf{E}_{\infty Y} \Rightarrow \mathsf{E}_{(x,Y)R}) \wedge (\forall x, Y : \mathsf{I}_{\infty x} \wedge \overline{\mathsf{E}}_{\infty Y} \Rightarrow \overline{\mathsf{E}}_{(x,Y)R}) \\
\Leftrightarrow\ & (\forall x, Y : [\mathsf{I}, \mathsf{E}]_{\infty(x,Y)} \Rightarrow \mathsf{E}^\smile{}_{R(x,Y)}) \wedge (\forall x, Y : [\mathsf{I}, \overline{\mathsf{E}}]_{\infty(x,Y)} \Rightarrow \overline{\mathsf{E}}^\smile{}_{R(x,Y)}) \\
\Leftrightarrow\ & (\mathsf{E}^\smile/[\mathsf{I}, \mathsf{E}])_{R\infty} \wedge (\overline{\mathsf{E}}^\smile/[\mathsf{I}, \overline{\mathsf{E}}])_{R\infty} \\
\Leftrightarrow\ & \exists x : (\mathsf{E}^\smile/[\mathsf{I}, \mathsf{E}])_{Rx} \wedge (\overline{\mathsf{E}}^\smile/[\mathsf{I}, \overline{\mathsf{E}}])_{Rx} \wedge x = \infty \\
\Leftrightarrow\ & \exists x, Y : ((\mathsf{E}^\smile/[\mathsf{I}, \mathsf{E}]) \cap (\overline{\mathsf{E}}^\smile/[\mathsf{I}, \overline{\mathsf{E}}]))_{Rx} \wedge \mathsf{N}_{xY} \wedge \mathsf{T}_{Y\star} \\
\Leftrightarrow\ & (((\mathsf{E}^\smile/[\mathsf{I}, \mathsf{E}]) \cap (\overline{\mathsf{E}}^\smile/[\mathsf{I}, \overline{\mathsf{E}}]))\mathsf{NT})_{R\star}
\end{aligned}
$$

The ranges of quantified variables are $x \in A$ and $Y \subseteq A$. $\square$

We apply the above construction again to enumerate all strict, up-closed multirelations as the columns of a relation:

$$\mathsf{E}\,\mathrm{inj}\Big(\overline{(\mathsf{E}^\smile(\mathsf{I}\|\mathsf{S}) \cap \overline{\mathsf{E}}^\smile)}\mathsf{T} \cap ((\mathsf{E}^\smile/[\mathsf{I},\mathsf{E}]) \cap (\overline{\mathsf{E}}^\smile/[\mathsf{I},\overline{\mathsf{E}}]))\mathsf{N}\mathsf{T}\Big)^\smile$$

For example, the following relation is obtained for $A = B = \{\infty, 1\}$:



Columns 1, 3, 4, 6 contain the multirelations $\mathsf{N} \cap \mathsf{L}$, L, E, $\overline{\mathsf{N}} \cup \mathsf{L}$, respectively. The matrix comprises columns 13–18 of the matrix in Section 5.1.

The following result shows that many multirelational operations preserve strictness.

Theorem 13.

1. *L, E, $\mathsf{N} \cap \mathsf{L}$ and $\overline{\mathsf{N}} \cup \mathsf{L}$ are strict.*
2. *$Q\,;R$ is strict if Q and R are strict and Q is up-closed.*
3. *R^d is strict if R is strict.*
4. *$\bigcup_{i\in I} R_i$ and $\bigcap_{i\in I} R_i$ are strict if R_i is strict for each $i \in I$ and $I \neq \emptyset$.*

Proof.

1. These claims follow from Theorems 9.13 and 9.9 and $\mathsf{N} \cap (\mathsf{N} \cap \mathsf{L}) = \mathsf{N} \cap \mathsf{L}$ and $\mathsf{N} \cap (\overline{\mathsf{N}} \cup \mathsf{L}) = \mathsf{N} \cap \mathsf{L}$.
2. $\mathsf{L}\,;(Q\,;R) = (\mathsf{L}\,;Q)\,;R = \mathsf{L}\,;R = \mathsf{L}$ using Theorem 7.2 as Q is up-closed and that Q and R are strict.
3. $\mathsf{L}\,;R^d = \mathsf{L}^d\,;R^d = (\mathsf{L}\,;R)^d = \mathsf{L}^d = \mathsf{L}$ using Theorems 9.10, 9.4 and 7.3 and that R is strict.
4. $\mathsf{N} \cap \bigcup_{i\in I} R_i = \bigcup_{i\in I}(\mathsf{N} \cap R_i) = \bigcup_{i\in I}(\mathsf{N} \cap \mathsf{L}) = \mathsf{N} \cap \mathsf{L}$ using that R_i is strict. The assumption $I \neq \emptyset$ is needed for the last equality. Moreover $\mathsf{N} \cap \bigcap_{i\in I} R_i = \bigcap_{i\in I}(\mathsf{N} \cap R_i) = \bigcap_{i\in I}(\mathsf{N} \cap \mathsf{L}) = \mathsf{N} \cap \mathsf{L}$ using that $\cap$ distributes over non-empty intersections and that R_i is strict. $\square$

A strict multirelation is obtained from a multirelation R by applying the function $s(R) = (\mathsf{N} \cap \mathsf{L}) \cup (\overline{\mathsf{N}} \cap R)$. The following result shows properties of s.

Theorem 14.

1. *Both the image of s and the fixpoints of s are precisely the strict multirelations.*
2. *s is $\subseteq$-isotone and idempotent.*
3. *$s(\mathsf{O}) = s(\mathsf{N}) = \mathsf{N} \cap \mathsf{L}$ and $s(\mathsf{T}) = \overline{\mathsf{N}} \cup \mathsf{L}$ and $s(\mathsf{L}) = \mathsf{L}$ and $s(\mathsf{E}) = \mathsf{E}$.*

Proof.

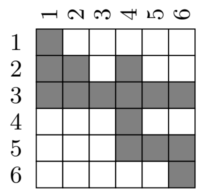
1. $\mathsf{N} \cap s(R) = \mathsf{N} \cap ((\mathsf{N} \cap \mathsf{L}) \cup (\overline{\mathsf{N}} \cap R)) = (\mathsf{N} \cap \mathsf{N} \cap \mathsf{L}) \cup (\mathsf{N} \cap \overline{\mathsf{N}} \cap R) = (\mathsf{N} \cap \mathsf{L}) \cup \mathsf{O} = \mathsf{N} \cap \mathsf{L}$ shows that $s(R)$ is strict. A strict multirelation R is a fixpoint of s since

   $$s(R) = (\mathsf{N} \cap \mathsf{L}) \cup (\overline{\mathsf{N}} \cap R) = (\mathsf{N} \cap R) \cup (\overline{\mathsf{N}} \cap R) = (\mathsf{N} \cup \overline{\mathsf{N}}) \cap R = \mathsf{T} \cap R = R$$

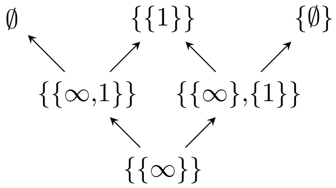
   A fixpoint of s is clearly in the image of s.
2. s is $\subseteq$-isotone since $\cap$ and $\cup$ are $\subseteq$-isotone. By the preceding claim, $s(s(R)) = s(R)$ since $s(R)$ is strict.
3. These claims follow by simple calculations using the definition of s and Theorem 9.9. $\square$

5.3. Deriving an approximation order (continued)

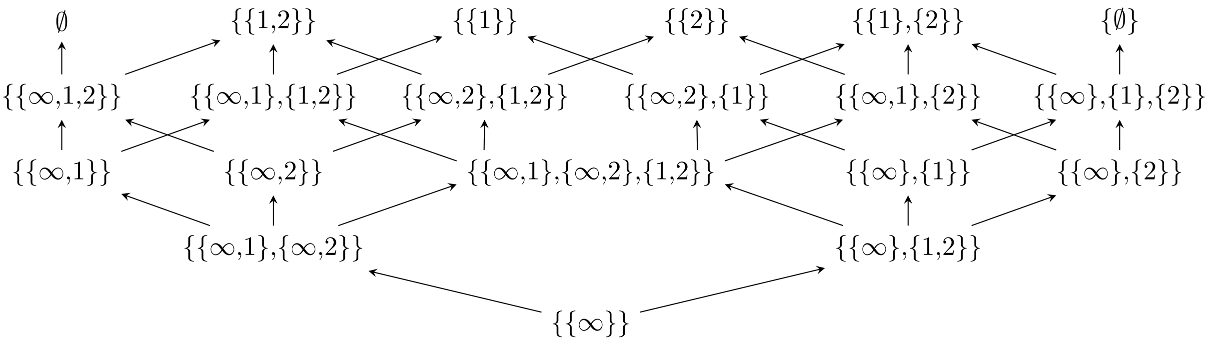
We apply the algorithm of Section 5.1 for $A = B = \{\infty, 1\}$, but this time restricted to the strict, up-closed multirelations. The result is



Rows and columns are numbered according to the enumeration produced in Section 5.2. RelView easily confirms that this relation is antisymmetric, hence a partial order. Its Hasse-diagram computed by RelView is shown below. Every node is labelled with the image of state 1 in the corresponding multirelation, that is, with the outcome for input 1 to be interpreted as in Section 3.4. In each set, only the ⊆-minimal subsets are given; their upward closure is not included:



Applying the algorithm of Section 5.1 for $A = B = \{\infty, 1, 2\}$ also yields a partial order. Its Hasse-diagram is:



Again the upward closure is omitted: for example, $\{\emptyset\}$ represents the set 2^A and $\{\{\infty, 1\}, \{2\}\}$ represents the set $\{\{\infty, 1\}, \{\infty, 2\}, \{\infty, 1, 2\}, \{2\}, \{1, 2\}\}$.

Denote the above order by $\sqsubseteq_2$. We observe the following about two sets related by this order $Xs \sqsubseteq_2 Ys$:

- If Xs contains a set X that does not contain ∞, then also Ys contains X.
- If Ys contains a set Y, then Xs contains Y or a set X such that $\infty \in X$ and Y contains all elements of X except perhaps ∞.

The underlying interpretation is that ∞ in a set in Xs may be replaced by any number of elements when going from Xs to Ys; a set in Xs that contains ∞ may be omitted in Ys and a set in Xs that does not contain ∞ must remain in Ys. Our observations hold for $A = \{\infty, 1\}$ and for $A = \{\infty, 1, 2\}$, but we expect that they hold for arbitrary A. This generalises the Egli-Milner order to sets of sets.

The above observations are formalised by

$$\begin{aligned} Xs \sqsubseteq_2 Ys \Leftrightarrow {} & (\forall X \in Xs : \infty \notin X \Rightarrow X \in Ys) \wedge \\ & (\forall Y \in Ys : \exists X \in Xs : X = Y \vee (\infty \in X \wedge X \subseteq Y \cup \{\infty\})) \end{aligned}$$

The approximation order on strict, up-closed multirelations is $\sqsubseteq_2$ applied pointwise to each state:

$$Q \sqsubseteq R \Leftrightarrow \forall x \in A : Q(x) \sqsubseteq_2 R(x)$$

Here $Q(x) = \{Y \mid Q_{xY}\}$ is the image of x under Q and similarly $R(x)$ is the image of x under R. The following result expresses $\sqsubseteq$ in terms of relations.

Theorem 15. *$Q \sqsubseteq R \Leftrightarrow R \cap \mathsf{L} \subseteq Q \subseteq R \cup \mathsf{L}$ if Q and R are up-closed.*

PROOF. $Q \sqsubseteq R$ is equivalent to

$$\begin{aligned}\forall x \in A : (&\forall X \in Q(x) : \infty \notin X \Rightarrow X \in R(x)) \wedge \\ &(\forall Y \in R(x) : \exists X \in Q(x) : X = Y \vee (\infty \in X \wedge X \subseteq Y \cup \{\infty\}))\end{aligned}$$

The first term is equivalently transformed by

$$\begin{aligned}&\forall x : \forall X \in Q(x) : \infty \notin X \Rightarrow X \in R(x) \\ \Leftrightarrow\ &\forall x, X : Q_{xX} \Rightarrow (\infty \notin X \Rightarrow R_{xX}) \\ \Leftrightarrow\ &\forall x, X : Q_{xX} \Rightarrow (\infty \in X \vee R_{xX}) \\ \Leftrightarrow\ &\forall x, X : Q_{xX} \Rightarrow (\mathsf{L}_{xX} \vee R_{xX}) \\ \Leftrightarrow\ &\forall x, X : Q_{xX} \Rightarrow (R \cup \mathsf{L})_{xX} \\ \Leftrightarrow\ &Q \subseteq R \cup \mathsf{L}\end{aligned}$$

The ranges of quantified variables are $x \in A$ and $X \subseteq A$. The second term is equivalently transformed by

$$\begin{aligned}&\forall x : \forall Y \in R(x) : \exists X \in Q(x) : X = Y \vee (\infty \in X \wedge X \subseteq Y \cup \{\infty\}) \\ \Leftrightarrow\ &\forall x, Y : R_{xY} \Rightarrow \exists X : Q_{xX} \wedge (X = Y \vee (\infty \in X \wedge X \subseteq Y \cup \{\infty\})) \\ \Leftrightarrow\ &\forall x, Y : R_{xY} \Rightarrow \exists X : Q_{xX} \wedge (\mathsf{I}_{XY} \vee (\infty \in X \wedge \exists Z : X \subseteq Z \wedge Z = Y \cup \{\infty\})) \\ \Leftrightarrow\ &\forall x, Y : R_{xY} \Rightarrow \exists X : Q_{xX} \wedge (\mathsf{I}_{XY} \vee ((\exists y : \mathsf{L}_{yX}) \wedge \exists Z : \mathsf{S}_{XZ} \wedge \mathsf{K}_{ZY})) \\ \Leftrightarrow\ &\forall x, Y : R_{xY} \Rightarrow \exists X : Q_{xX} \wedge (\mathsf{I}_{XY} \vee ((\mathsf{TL})_{YX} \wedge (\mathsf{SK})_{XY})) \\ \Leftrightarrow\ &\forall x, Y : R_{xY} \Rightarrow \exists X : Q_{xX} \wedge (\mathsf{I}_{XY} \vee ((\mathsf{TL})^{\smile} \cap \mathsf{SK})_{XY}) \\ \Leftrightarrow\ &\forall x, Y : R_{xY} \Rightarrow \exists X : Q_{xX} \wedge (\mathsf{I} \cup ((\mathsf{TL})^{\smile} \cap \mathsf{SK}))_{XY} \\ \Leftrightarrow\ &\forall x, Y : R_{xY} \Rightarrow (Q(\mathsf{I} \cup ((\mathsf{TL})^{\smile} \cap \mathsf{SK})))_{xY} \\ \Leftrightarrow\ &R \subseteq Q(\mathsf{I} \cup ((\mathsf{TL})^{\smile} \cap \mathsf{SK}))\end{aligned}$$

The ranges of quantified variables are $x, y \in A$ and $X, Y \subseteq A$. The result simplifies to

$$\begin{aligned}R &\subseteq Q(\mathsf{I} \cup ((\mathsf{TL})^{\smile} \cap \mathsf{SK})) = Q\mathsf{I} \cup Q(\mathsf{L}^{\smile}\mathsf{T} \cap \mathsf{SK}) = Q \cup (Q \cap \mathsf{TL})\mathsf{SK} = Q \cup (Q \cap \mathsf{TL})\mathsf{K} \\ &= Q \cup Q(\mathsf{L}^{\smile}\mathsf{T} \cap \mathsf{K}) = Q \cup Q\mathsf{K} = Q\mathsf{I} \cup Q\mathsf{SK} = Q(\mathsf{I} \cup \mathsf{SK}) = Q\mathsf{SK} = Q\mathsf{K}\end{aligned}$$

using that Q is up-closed and Theorems 9.4, 7.6, 7.7, 10.2 and 10.3. This is equivalent to $R\mathsf{K}^{\smile} \subseteq Q$ using Theorem 10.1, and thus to $R \cap \mathsf{L} \subseteq Q$ by Theorem 10.6 since R is up-closed. □

5.4. Median

Given the characterisation of Theorem 15, we find that $\sqsubseteq$ is the semilattice order induced by the median operation. By Theorems 7.7 and 13.4 the strict, up-closed multirelations are closed under $\cup$ and $\cap$ and thus form a distributive lattice. Hence we can look at the ternary median operation [9, 6, 5]:

$$(P, Q, R) = (P \cap Q) \cup (Q \cap R) \cup (R \cap P)$$

It is self-dual and a collection of its symmetries is given in [19]. Consider the instance

$$Q \sqcap R = (\mathsf{L}, Q, R) = (\mathsf{L} \cap Q) \cup (Q \cap R) \cup (R \cap \mathsf{L})$$

Our previous result [10, Proposition 20] specialises to multirelations as follows.

Theorem 16. *The strict, up-closed multirelations form a semilattice with meet $\sqcap$, semilattice order $\sqsubseteq$ and $\sqsubseteq$-least element L. The following hold:*

1. $Q \sqsubseteq R \Leftrightarrow Q = Q \sqcap R$.
2. $\cap$, $\cup$ *and* $\sqcap$ *distribute over each other.*
3. $Q \subseteq R \Leftrightarrow Q \cap \mathsf{L} \subseteq R \wedge Q \subseteq R \cup \mathsf{L}$.
4. $Q \sqcap R = (R \cap \mathsf{L}) \cup Q = R \cap (\mathsf{L} \cup Q)$ *if* $Q \subseteq R$.

Further results based on the median instantiate to multirelations. They include representations of $\sqsubseteq$-least (pre-)fixpoints in terms of $\subseteq$-least (pre-)fixpoints and $\subseteq$-greatest (post-)fixpoints. We present [10, Theorem 24] as an example.

Theorem 17. *Let f be a $\subseteq$- and $\sqsubseteq$-isotone function on strict, up-closed multirelations. Let μf be the $\subseteq$-least fixpoint of f. Let νf be the $\subseteq$-greatest fixpoint of f. Then the $\sqsubseteq$-least fixpoint ξf of f exists and satisfies $\xi f = \mu f \sqcap \nu f = (\nu f \cap \mathsf{L}) \cup \mu f$.*

We prove the following new result, which also holds in arbitrary pointed distributive lattices.

Theorem 18. *$\sqsubseteq$ is the $\subseteq$-least partial order with least element L and isotone operations $\cup$ and $\cap$.*

Proof. $\cup$ and $\cap$ are $\sqsubseteq$-isotone by the distributivity properties in Theorem 16. Let $\preceq$ be a partial order with least element L such that $\cup$ and $\cap$ are $\preceq$-isotone. Let $Q \sqsubseteq R$, whence $R \cap \mathsf{L} \subseteq Q \subseteq R \cup \mathsf{L}$. Hence $(R \cap \mathsf{L}) \cup Q = Q = Q \cap (R \cup \mathsf{L})$. First, $\mathsf{L} \preceq R$ implies $R \cup \mathsf{L} \preceq R \cup R = R$, and therefore $Q = Q \cap (R \cup \mathsf{L}) \preceq Q \cap R$. Second, $\mathsf{L} \preceq R$ implies $R \cap \mathsf{L} \preceq R \cap R = R$, hence $Q = Q \cup (R \cap \mathsf{L}) \preceq Q \cup R$, and therefore $Q \cap R \preceq (Q \cup R) \cap R = R$. Together, $Q \preceq Q \cap R \preceq R$. ☐

The following result shows that other operations on multirelations are $\sqsubseteq$-isotone, too.

Theorem 19. *$\cdot^d$ and $\,;\,$ are $\sqsubseteq$-isotone on the strict, up-closed multirelations.*

Proof. Let $Q \sqsubseteq R$, whence $R \cap \mathsf{L} \subseteq Q \subseteq R \cup \mathsf{L}$. First,

$$R^d \cap \mathsf{L} = R^d \cap \mathsf{L}^d = (R \cup \mathsf{L})^d \subseteq Q^d \subseteq (R \cap \mathsf{L})^d = R^d \cup \mathsf{L}^d = R^d \cup \mathsf{L}$$

using Theorems 9.10, 5.7, 5.1 and 5.8. Hence $Q^d \sqsubseteq R^d$. Second,

$$(R \,;\, P) \cap \mathsf{L} = (R \,;\, P) \cap (\mathsf{L} \,;\, P) = (R \cap \mathsf{L}) \,;\, P \subseteq Q \,;\, P \subseteq (R \cup \mathsf{L}) \,;\, P = (R \,;\, P) \cup (\mathsf{L} \,;\, P) = (R \,;\, P) \cup \mathsf{L}$$

using that P is strict, that R is up-closed and Theorems 9.4, 7.8, 3.1 and 3.7. Hence $Q \,;\, P \sqsubseteq R \,;\, P$. Third,

$$\begin{aligned}(P \,;\, R) \cap \mathsf{L} &= P(\mathsf{E}\backslash R) \cap \mathsf{TL} = P((\mathsf{E}\backslash R) \cap \mathsf{TL}) \subseteq P((\mathsf{E}\backslash R) \cap (\mathsf{E}\backslash \mathsf{L})) = P(\mathsf{E}\backslash (R \cap \mathsf{L})) \\ &= P \,;\, (R \cap \mathsf{L}) \subseteq P \,;\, Q \subseteq P \,;\, (R \cup \mathsf{L}) = P(\mathsf{E}\backslash (R \cup \mathsf{L})) = P\overline{\mathsf{E}^\smile \overline{R \cup \mathsf{L}}} = P\overline{\mathsf{E}^\smile (\overline{R} \cap \overline{\mathsf{L}})} \\ &= P\overline{\mathsf{E}^\smile (\overline{R} \cap \mathsf{T}\overline{\mathsf{L}})} = P\overline{\mathsf{E}^\smile \overline{R} \cap \mathsf{T}\overline{\mathsf{L}}} = P(\overline{\mathsf{E}^\smile \overline{R}} \cup \overline{\mathsf{T}\overline{\mathsf{L}}}) = P(\mathsf{E}\backslash R) \cup P\overline{\mathsf{T}\overline{\mathsf{L}}} \subseteq (P \,;\, R) \cup \mathsf{L}\end{aligned}$$

using Theorems 9.5 and 9.6, (6), Theorems 3.1 and 9.5, and that $P\overline{\mathsf{T}\overline{\mathsf{L}}} \subseteq \mathsf{L}$ by a Schröder equivalence. Hence $P \,;\, Q \sqsubseteq P \,;\, R$. ☐

6. Conclusion

This paper shows that relations are useful for investigating multirelations and extending the underlying state space to represent infinite computations. Our development essentially benefited from exploring the approximation order with RelView, which led to the compact algebraic characterisation. In future work we will apply the given method to derive approximation orders for further computation models.

Acknowledgements

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Appendix. RelView implementation

```
sub(R)    = epsi(R) \ epsi(R).           { subset relation of size 2^dom(R) }
dual(R)   = -R * syq(epsi(R),-epsi(R)).  { dual of multirelation R }
comp(R,S) = R * (epsi(R) \ S).           { composition of multirelations R and S }

Upclosed(R)                              { enumerate all up-closed multirelations of size R }
  DECL Prod = PROD(R*R^,R^*R)
       P,E,V
  BEG
    P = p-1(Prod)
    E = epsi(P)
    V = -((E^ * par(I(R*R^),sub(R)) & -E^) * Ln1(P))
    RETURN E * inj(V)^                   { every column gives a multirelation as a vector }
  END

StrictUpclosed(N)                        { enumerate all strict, up-closed multirelations }
  DECL Prod = PROD(N*N^,N^*N)
       E,EP,U,S
  BEG
    E = epsi(N)
    EP = epsi(p-1(Prod))
    U = -dom(EP^ * par(I(N*N^),sub(N)) & -EP^)
    S = dom(((EP^ / [I(N*N^),E]) & (-EP^ / [I(N*N^),-E])) * N)
    RETURN EP * inj(U & S)^              { every column gives a multirelation as a vector }
  END

Loop(N) = L(N*N^) * (N & epsi(N)).       { multirelation representing the endless loop }

Strict(U,N)                              { select strict multirelations from up-closed ones }
  DECL LL
  BEG
    LL = Loop(N)
    RETURN U * inj((r2v(LL & N) \ U)^ & (U \ r2v(LL | -N)))^
  END

col_index(U,R) = syq(U,r2v(R) * L1n(U)). { point indexing multirelation R from columns of U }

isol(U,X,Y,Z) = col_index(U,comp(X,Z)) & col_index(U,comp(Y,Z))^.
isor(U,X,Y,Z) = col_index(U,comp(Z,X)) & col_index(U,comp(Z,Y))^.
isoj(U,X,Y,Z) = col_index(U,X | Z) & col_index(U,Y | Z)^.
isom(U,X,Y,Z) = col_index(U,X & Z) & col_index(U,Y & Z)^.
isod(U,X,Y)   = col_index(U,dual(X)) & col_index(U,dual(Y))^.

APX(U,N)                                 { least order satisfying constraints stated below }
  DECL LL,A,Aprev,Atodo,PA,X,Y,Utodo,PU,Z
  BEG
    LL = Loop(N)
    A = refl(trans(col_index(U,LL)))     { order has least element LL }
    Aprev = O(A)
```

```
    WHILE -eq(A,Aprev) DO                { order changed? }
      Aprev = A
      Atodo = A
      WHILE -empty(Atodo) DO             { process all pairs (X,Y) in current order }
        PA = atom(Atodo)
        Atodo = Atodo & -PA
        X = v2r(U * dom(PA),LL)
        Y = v2r(U * dom(PA^),LL)
        Utodo = Ln1(U^)
        WHILE -empty(Utodo) DO           { combine with all multirelations Z in U }
          PU = point(Utodo)
          Utodo = Utodo & -PU
          Z = v2r(U * PU,LL)             { multirelational operations are isotone }
          A = A | isol(U,X,Y,Z) | isor(U,X,Y,Z) | isoj(U,X,Y,Z) | isom(U,X,Y,Z) | isod(U,X,Y)
        OD
      OD
      A = refl(trans(A))
    OD
    RETURN A
  END

Hasse(A) = A & -I(A) & -((A & -I(A)) * (A & -I(A))).

APXrel(U,N)                              { order based on relational definition }
  DECL LL,A,Todo,P,Q,R                   { gives the same result as APX }
  BEG
    LL = Loop(N)
    A = O(U^ * U)
    Todo = L(A)
    WHILE -empty(Todo) DO                { process all pairs (Q,R) of multirelations in U }
      P = atom(Todo)
      Todo = Todo & -P
      Q = v2r(U * dom(P),LL)
      R = v2r(U * dom(P^),LL)            { evaluate relational definition for Q and R }
      IF incl(R & LL,Q) & incl(Q,R | LL) THEN
        A = A | P
      FI
    OD
    RETURN A
  END

{ The following functions are taken from RelView libraries. }

par(R,S)                                 { parallel composition of relations R and S }
  DECL ProdDom = PROD(R*R^,S*S^)
       ProdRan = PROD(R^*R,S^*S)
  BEG
    RETURN (p-1(ProdDom) * R * p-1(ProdRan)^) & (p-2(ProdDom) * S * p-2(ProdRan)^)
  END

r2v(R)                                   { transform relation R:A<->B to vector A*B<->1 }
  DECL Prod = PROD(R*R^,R^*R)
  BEG
    RETURN dom(p-1(Prod) * R & p-2(Prod))
  END
```

```
v2r(V,R)                                { transform vector V:A*B<->1 to relation A<->B }
  DECL Prod = PROD(R*R^,R^*R)           { the result has the same type as R }
  BEG
    RETURN p-1(Prod)^ * (p-2(Prod) & V * L1n(R))
  END
```

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