Typing Theorems of Omega Algebra

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Abstract

Typed omega algebras extend Kozen's typed Kleene algebras by an operation for infinite iteration in a similar way as Cohen's omega algebras extend Kleene algebras in the untyped case. Typing these algebras is motivated by non-square matrices in automata constructions and applications in program semantics.

For several reasons – the theory of untyped (Kleene or omega) algebras is well developed, results are easier to derive, and automation support is much better – it is beneficial to transfer theorems from the untyped algebras to their typed variants instead of constructing new proofs in the typed setting. Such a typing of theorems is facilitated by embedding typed algebras into their untyped variants.

Extending previous work, we show that a large class of theorems of 1-free omega algebras can be transferred to typed omega algebras. This covers every universal 1-free formula which does not contain the greatest element at the beginning of an expression in a negative occurrence of an equation. Moreover, the formulas may be infinitary.

Keywords: heterogeneous relations, Kleene algebra, matrix algebra, non-square matrices, omega algebra, 1-free expression, typed omega algebra, typing theorems

1. Introduction

Typed algebras have been investigated in various contexts, for example, heterogeneous relation algebras [20] for relations between different sets, typed Kleene algebras [15] for non-square matrices representing regular expressions, and allegories [7] for program development [3]. Each element of a typed algebra is endowed with a type, and operations of the algebra are partial, defined only for elements with matching types. The typed algebras differ as regards their operations – see [11] for a hierarchy of algebras based on ordered categories – comprising various combinations of join, meet, complement, converse, composition, Kleene star, domain and others. Typed Kleene algebras, in particular, support join, composition and the Kleene star for modelling choice, sequence and finite iteration which occur in many applications. Typed omega algebras [9] extend typed Kleene algebras by an operation for infinite iteration in a similar way as omega algebras [5] extend Kleene algebras [14] in the untyped case.

Motivation for using typed Kleene algebras comes from constructions related to automata; several examples follow. The language accepted by a finite automaton is $u^T A^* v$ where the vector u encodes the initial states, the matrix A the transitions and the vector v the final states of the automaton; hence u and v are non-square matrices [6]. Finite matrices over a Kleene algebra form a Kleene algebra, where the Kleene star of an $n \times n$ matrix is given by partitioning it into smaller matrices [6]:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} e^* & a^* b f^* \\ d^* c e^* & f^* \end{pmatrix} \qquad \text{with} \qquad \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} a + b d^* c \\ d + c a^* b \end{pmatrix} .$$

The submatrices a and d are square, but in general b and c are not. Calculations involving b or c, such as proving correctness of this star rule for matrices, work in typed Kleene algebras. Simulation properties, such as $ax = xb \Rightarrow a^* x = x b^*$, are used in the completeness result for Kleene algebras also with non-square matrices x [14, 15]. In these contexts the omega operation may be used to model infinite executions of automata. In [9] we apply typed omega algebras for calculating the omega operation for matrix representations of programs in general correctness.

In this paper we discuss the problem whether formulas which are valid in (untyped) omega algebras also hold in typed omega algebras, under all possible typings. Note that the converse question is trivial as an untyped algebra can be regarded as a typed algebra with a single type. For Kleene algebras, the problem has been treated in [15] with the following results. The validity of every universal 1-free formula is preserved, where 1-free formulas omit the constant 1 and use the non-empty iteration $^+$ instead of the Kleene star *. The universal formula $0 = 1 \Rightarrow a = b$ holds in Kleene algebras, but not in typed Kleene algebras in which one type has only one element (hence $0 = 1$ there) and another type has more (hence $a \neq b$ there, as the types of 0, 1 and a, b are unrelated). Axioms might be added to prevent this counterexample: for example, the formula holds in heterogeneous relation algebra. But another counterexample $1 = \top \Rightarrow 1 = \top$ shows that there is no simple way around 1-freeness (in a typed instance of this formula, the types of 1, $\top$ in the antecedent and in the consequent may differ). Because the greatest element $\top$ is available in typed omega algebras, we maintain the restriction to 1-free formulas. Further restriction to $\top$-free formulas is not necessary: for example, $0 = \top \Rightarrow a = b$ holds in typed omega algebras with $\top = \top\top$ and $c \leq \top$ for every c.

We show that a large class of universal 1-free formulas can be transferred from omega algebras to typed omega algebras. Two ways to prove such a result are presented in [15]: a proof-theoretic one and a model-theoretic one. The first derives typed proofs from untyped proofs, which is difficult already for Horn formulas; restricted to equations in Kleene algebras and residuated lattices, this approach is used in [19]. The second constructs embeddings of typed algebras in untyped algebras; restricted to 1-free Kleene algebras, it covers arbitrary universal formulas [15]. We use the second, model-theoretic approach.

Reasoning tools particularly benefit from results that generalise untyped theorems to typed theorems. While types of expressions can be encoded in many systems, few directly support reasoning about typed formulas. For example, implementations of heterogeneous relation algebras and similar structures are described in [2, 18, 10, 1, 12]; proofs are typically interactive with limited automation capabilities. Reduction of typed formulas to untyped formulas enables the use of numerous readily available automated theorem provers and decision procedures for untyped algebras to derive results for the typed setting [4, 19].

In Section 2 we recapitulate Kleene algebras, omega algebras, their 1-free and typed variants and the construction of matrices over typed 1-free omega algebras. Section 3 recalls our previous results [9], which embed finitely typed 1-free omega algebras into 1-free omega algebras, and the latter into omega algebras. Thereby restricted forms of universal statements are valid in the untyped setting if and only if they are valid in the typed setting. The embeddings require different subsets of axioms, and some do not preserve the greatest element $\top$.

Sections 4–6 contain the contributions of this paper which extend our previous results. On the negative side, we show in Section 4 that an attempt to preserve $\top$ by making a simple change to our embeddings works only in very special cases. It turns out that a modification is actually unnecessary. Namely, on the positive side, we show in Section 6 that our embeddings in fact preserve $\top$ in many contexts, whence a large class of theorems with $\top$ can be transferred from the untyped to the typed setting. This includes all theorems in which expressions in negated equations do not 'begin with $\top$', in the sense of the first transition when regarded as an automaton. Section 5 generalises our results to infinitely typed 1-free omega algebras. As a consequence, preservation of validity extends to infinitary formulas.

2. Typed and 1-Free Omega Algebras

We recapitulate the axioms for (typed) (1-free) Kleene and omega algebras and the construction of matrix algebras. Our exposition is based on [9].

2.1. Omega Algebra

We start with the axioms of semirings, Kleene algebras and omega algebras. An idempotent semiring is a structure $(S, +, \cdot, 0, 1)$ that satisfies the following axioms:

$$\begin{aligned}a+(b+c) &= (a+b)+c & a(b+c) &= ab+ac & a(bc) &= (ab)c\\ a+b &= b+a & (a+b)c &= ac+bc & 1a &= a\\ a+a &= a & 0a &= 0 & a1 &= a\\ a+0 &= a & a0 &= 0 & &\end{aligned}$$

The operation $\cdot$ has higher precedence than $+$ and is frequently omitted by writing ab instead of $a \cdot b$. By $a \leq b \Leftrightarrow a+b=b$ we obtain the partial order $\leq$ on S with join $+$ and least element 0. The operations $+$ and $\cdot$ are $\leq$-isotone.

A Kleene algebra [14] is a structure $(S, +, \cdot, ^*, 0, 1)$ such that $(S, +, \cdot, 0, 1)$ is an idempotent semiring and the following axioms hold:

$$\begin{aligned}1+aa^* &= a^* & b+ac \leq c &\Rightarrow a^*b \leq c\\ 1+a^*a &= a^* & b+ca \leq c &\Rightarrow ba^* \leq c\end{aligned}$$

The operation * is $\leq$-isotone and has highest precedence. Every Kleene algebra has the non-empty iteration $a^+ =_{\text{def}} aa^* = a^*a$. It satisfies $a^* = 1 + a^+$ and

$$\begin{aligned}a+aa^+ &= a^+ & b+ac \leq c &\Rightarrow a^+b \leq c\\ a+a^+a &= a^+ & b+ca \leq c &\Rightarrow ba^+ \leq c\end{aligned}$$

The operation $^+$ is $\leq$-isotone and has the same precedence as *.

An omega algebra [5] is a structure $(S, +, \cdot, ^*, ^\omega, 0, 1)$ such that $(S, +, \cdot, ^*, 0, 1)$ is a Kleene algebra and the following axioms hold:

$$aa^\omega = a^\omega \qquad c \leq ac+b \Rightarrow c \leq a^\omega + a^*b$$

The operation $^\omega$ is $\leq$-isotone and has the same precedence as *. Every omega algebra has a $\leq$-greatest element $\top =_{\text{def}} 1^\omega$. It satisfies

$$\begin{aligned}a^\omega\top &= a^\omega & a &\leq a\top & \top &= \top\top\\ a &\leq \top & a &\leq \top a & &\end{aligned}$$

We call those axioms of Kleene and omega algebra, which are implications, induction axioms.

2.2. 1-Free Omega Algebra

We continue with the axioms of 1-free Kleene algebras and 1-free omega algebras. The restriction to 1-free algebras enables the transfer of universal formulas from the untyped to the typed setting.

A 1-free Kleene algebra [15] is a structure $(S, +, \cdot, ^+, 0)$ that satisfies the idempotent semiring axioms without 1, that is,

$$\begin{aligned}a+(b+c) &= (a+b)+c & a(b+c) &= ab+ac & a(bc) &= (ab)c\\ a+b &= b+a & (a+b)c &= ac+bc & &\\ a+a &= a & 0a &= 0 & &\\ a+0 &= a & a0 &= 0 & &\end{aligned}$$

and the laws about $^+$ mentioned above

$$\begin{aligned}a+aa^+ &= a^+ & b+ac \leq c &\Rightarrow a^+b \leq c\\ a+a^+a &= a^+ & b+ca \leq c &\Rightarrow ba^+ \leq c\end{aligned}$$

which replace the *-axioms. An equivalent structure is obtained by replacing the implications with

$$ac \le c \Rightarrow a^+c \le c$$
$$ca \le c \Rightarrow ca^+ \le c$$

It follows that the operation $^+$ is $\le$-isotone.

Every Kleene algebra extended by $a^+ = aa^*$ is a 1-free Kleene algebra. Every 1-free Kleene algebra with an element 1 such that $1a = a = a1$ and extended by $a^* = 1 + a^+$ is a Kleene algebra.

A 1-free omega algebra [9] is a structure $(S, +, \cdot, ^+, ^\omega, 0, \top)$ such that $(S, +, \cdot, ^+, 0)$ is a 1-free Kleene algebra and the following axioms hold:

$$aa^\omega = a^\omega \qquad\qquad c \le ac + b \Rightarrow c \le a^\omega\top + a^+b + b$$

The operation $^\omega$ is not $\le$-isotone in general, but $a \le b$ implies both $a^\omega \le b^\omega\top$ and $a^\omega\top \le b^\omega\top$.

The term $a^\omega\top$ replaces a^ω in the induction axiom to prepare it for the typed setting. Moreover, we discuss the following axioms about $^\omega$ and $\top$:

$$\begin{array}{llllll} (\top1) & a^\omega\top = a^\omega & (\top3) & a \le a\top & (\top5) & \top = \top\top \\ (\top2) & a \le \top & (\top4) & a \le \top a & & \end{array}$$

We explicitly state whenever they are used in addition to the axioms of 1-free omega algebra. Except for $(\top5)$, which follows from $(\top2)$ and either $(\top3)$ or $(\top4)$, these axioms are independent of each other and the axioms of 1-free omega algebra, as counterexamples generated by Mace4 witness.

Every omega algebra extended by $a^+ = aa^*$ is a 1-free omega algebra. Every 1-free omega algebra with an element 1 such that $1a = a = a1$ is an omega algebra, when extended by $a^* = 1 + a^+$ and $a^\omega = a^\Omega\top$, where a^Ω denotes the operation in the 1-free omega algebra; with $(\top1)$ we can take $a^\omega = a^\Omega$.

To improve readability, we use the * notation also in 1-free algebras to abbreviate terms of the form

$$\begin{array}{ll} a^*b = a^+b + b & ab^*c = ab^+c + ac \\ ba^* = ba^+ + b & a^*bc^* = a^+bc^+ + a^+b + bc^+ + b \end{array}$$

and similar ones, where * occurs in products with at least one factor not having the form a^*. For example, the omega induction axiom becomes $c \le ac + b \Rightarrow c \le a^\omega\top + a^*b$. Due to the semiring axioms, calculations using this notation work as expected. In such contexts * is $\le$-isotone and the star induction axioms hold.

2.3. Typed 1-Free Omega Algebra

We use the mechanism for typing described in [15]. In particular, we assume a set T of pretypes $s, t, u, v, \ldots$ and obtain the set T^2 of types denoted as $s \to t$. The type of an expression a of omega algebra is denoted by $a : s \to t$ and can be derived using a type calculus with the rules

$$\frac{a, b : s \to t}{a + b : s \to t} \qquad \frac{a : s \to t \qquad b : t \to u}{ab : s \to u} \qquad \frac{a : s \to s}{a^*, a^+, a^\omega : s \to s} \qquad \begin{array}{c} 0, \top : s \to t \\ 1 : s \to s \end{array}$$

The rules for $^\omega$ and $\top$ are added to those of typed Kleene algebras. Expressions a and b in an equation $a = b$ must have the same type. We also write a_{st} to make clear that a has type $s \to t$.

For example, finite heterogeneous relations are modelled by letting T be the natural numbers. Then $a : s \to t$ denotes that a is a Boolean matrix with s rows and t columns. See [15] for further details about the typing mechanism.

A typed Kleene algebra (with pretype set T) is a set S of typed elements $a : s \to t$ $(s, t \in T)$ with polymorphic operations $+$, $\cdot$, *, 0 and 1, typed according to the above inference rules, satisfying all well-typed instances of the Kleene algebra axioms.

Typed 1-free Kleene algebras and typed 1-free omega algebras are defined similarly, using all well-typed instances of the respective axioms in Section 2.2. All well-typed instances of a selection of $(\top1)$–$(\top5)$ may be considered as well. In the typed setting, $(\top5)$ is independent of the remaining axioms and $(\top1)$–$(\top4)$.

For a typed omega algebra we use all well-typed instances of the omega algebra axioms, except for omega induction, which we replace by the omega induction axiom of 1-free omega algebra $c \le ac+b \Rightarrow c \le a^\omega\top+a^*b$. With ($\top$1) this yields all well-typed instances of the original omega induction axiom; whether the converse holds is unknown. Typed (1-free) omega algebras have been introduced in [9].

A finitely typed algebra is one with finite T. We denote the set of elements with type $s \to t$ in a typed structure S by S_{st}. An untyped formula is valid in S if all its well-typed instances hold.

Every typed Kleene algebra is a typed 1-free Kleene algebra, when extended by $a_{ss}^+ = a_{ss}a_{ss}^*$ for each $s \in T$. Every typed 1-free Kleene algebra with elements 1_{ss} for each $s \in T$ such that $1_{ss}a_{st} = a_{st} = a_{st}1_{tt}$ for each $s, t \in T$ is a typed Kleene algebra, when extended by $a_{ss}^* = 1_{ss} + a_{ss}^+$ for each $s \in T$. Corresponding statements hold for typed omega algebras and typed 1-free omega algebras as both share the omega axioms.

We remark that the axiom ($\top$2) establishes $\top : s \to t$ as the greatest element of type $s \to t$. As in heterogeneous relation algebra, each type has its own greatest element. In the untyped setting, being the greatest element is the main property of $\top$. In the typed setting, emphasis should be on its property to cause a type cast, that is, a change of types: from $a : s \to t$ we obtain the element $a\top$ of type $s \to u$ by composing with $\top : t \to u$. Thus ($\top$5) decomposes a type cast effected by $\top : s \to u$ into a sequence of two type casts effected by $\top : s \to t$ and $\top : t \to u$. It is this type changing capacity which is used in the omega induction axiom. This ensures that $a^\omega\top$ is compatible with a^*b also if $b : s \to t$ with $s \neq t$.

Examples of typed omega algebras are the finite heterogeneous relations mentioned above, the algebras of finite matrices we describe next and particular algebras constructed to represent general correctness models of programs [9].

2.4. Matrices

We recapitulate how to obtain an algebra of finite matrices over typed omega algebras by lifting the underlying structure. Fix a typed 1-free omega algebra S with (not necessarily finite) pretype set T. We construct a typed 1-free omega algebra of finite matrices whose entries are elements of S. The pretypes of this matrix algebra are the finite sequences over T. Let $s_1, \dots, s_m \in T^m$ and $t_1, \dots, t_n \in T^n$ be pretypes, then a matrix has type $s_1, \dots, s_m \to t_1, \dots, t_n$ if and only if its size is $m \times n$ and, for each $1 \le i \le m$ and $1 \le j \le n$, the entry in row i and column j has type $s_i \to t_j$.

The operations $+$, $\cdot$, 0 and $\top$ are, as usual, the componentwise sum, the matrix product, the 0- and the $\top$-matrix, respectively. The non-empty iteration $^+$ is defined by $(a)^+ = (a^+)$ for 1×1 matrices and, inductively,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^+ = \begin{pmatrix} e^+ & a^*bf^* \\ d^*ce^* & f^+ \end{pmatrix} \qquad \text{with} \qquad \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} a+bd^*c \\ d+ca^*b \end{pmatrix}.$$

This is derived by $A^+ = AA^*$ from the usual matrix * of [6]. The infinite iteration $^\omega$ is given by $(a)^\omega = (a^\omega)$ for size 1×1 and, inductively,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\omega = \begin{pmatrix} e^\omega & a^*bf^\omega \\ d^*ce^\omega & f^\omega \end{pmatrix}\begin{pmatrix} \top & \top \\ \top & \top \end{pmatrix} \qquad \text{with} \qquad \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} a+bd^*c \\ d+ca^*b \end{pmatrix}.$$

By composing with the typed $\top$-matrix we obtain a matrix whose columns are not identical, as in the untyped case [17], but have their types adjusted. In [9] we prove the following consequences.

Theorem 1. *The finite matrices over a typed 1-free omega algebra form a typed 1-free omega algebra. Each of the axioms* ($\top$1)–($\top$5) *is preserved.*

Corollary 2. *The $n \times n$ matrices with fixed type over a typed 1-free omega algebra form a 1-free omega algebra. Each of the axioms* ($\top$1)–($\top$5) *is preserved.*

3. Embedding Omega Algebras and Typing Theorems

By embedding typed omega algebras into omega algebras we can transfer theorems from the untyped to the typed setting. This way, the typed setting benefits from existing theorems, simpler (untyped) proofs of new theorems, and automated theorem provers which have no notion of types. We first recapitulate the existing results about typed omega algebras and then summarise our extensions, the details of which we present in Sections 4–6.

In [9] we prove the following results about

- embedding 1-free omega algebras into omega algebras (Theorems 3 and 5),
- embedding finitely typed 1-free omega algebras into 1-free omega algebras (Theorem 7) and
- preserving the validity of statements with universally quantified variables (Corollaries 4, 6, 8 and 9).

They extend corresponding results of [15] for Kleene algebras.

Theorem 3. *Every 1-free omega algebra satisfying* ($\top$1) *and* ($\top$2) *can be embedded into an omega algebra, except that the embedding need not preserve* $\top$.

Corollary 4. *A universal formula using only the operations* $+$, $\cdot$, $^+$, $^\omega$, 0 *is valid in omega algebra if and only if it is valid in 1-free omega algebra with* ($\top$1) *and* ($\top$2).

Theorem 5. *Every 1-free omega algebra satisfying* ($\top$1)–($\top$4) *can be embedded into an omega algebra.*

Corollary 6. *A universal formula of 1-free omega algebra is valid in omega algebra if and only if it is valid in 1-free omega algebra with* ($\top$1)–($\top$4).

Consider a typed 1-free omega algebra S based on a set of n pretypes T, and arrange the pretypes in a fixed sequence $(t_i) \in T^n$. By Corollary 2, the $n \times n$ matrices with type $(t_i) \to (t_i)$ form a 1-free omega algebra. Theorem 7 embeds S into this matrix algebra using the following mapping h:

$$h(a_{st})_{uv} =_{\mathrm{def}} \begin{cases} a_{st} & \text{if } u = s \text{ and } v = t \\ a_{st}\top_{tv} & \text{if } u = s \text{ and } v \neq t \\ 0_{uv} & \text{if } u \neq s \end{cases}$$

Thus the element $a : s \to t$ is mapped to a matrix with a in row s and column t, with $a\top$ in any other column of row s, and 0 in any other row. Note that a typed embedding is required to be injective for each type, but may map elements of distinct types to the same element [16].

Theorem 7. *A finitely typed 1-free omega algebra satisfying* ($\top$1) *is embedded by* h *into a 1-free omega algebra, except that* h *need not preserve* $\top$. *Each of the axioms* ($\top$1)–($\top$5) *is preserved.*

Corollary 8. *A universal formula using only the operations* $+$, $\cdot$, $^+$, $^\omega$, 0 *is valid in 1-free omega algebra with* ($\top$1) *if and only if it is valid in typed 1-free omega algebra with* ($\top$1).

Corollary 9. *A universal formula using only the operations* $+$, $\cdot$, $^+$, $^\omega$, 0 *is valid in omega algebra if and only if it is valid in typed 1-free omega algebra with* ($\top$1) *and* ($\top$2).

These results leave open the case of formulas with $\top$. In Section 4 we show that a simple modification of the embedding h to preserve $\top$ does not solve this issue. In Section 6 we show that h in fact preserves $\top$ in various contexts, whence the validity of a large class of formulas with $\top$ is preserved. Among others, this covers all formulas in which expressions in negated equations do not 'begin with $\top$'. In Section 5 we generalise the previous results to infinitely typed algebras. We can thus embed all typed 1-free omega algebras and preserve the validity of infinitary formulas.

4. Typing Omega Algebras with $\top$: A Negative Result

To preserve $\top$ in addition to the other operations of 1-free omega algebra, it is tempting to modify the embedding h of Theorem 7 to the new mapping $\hbar$ defined in the following form using certain constants c_{uv} discussed below:

$$\hbar(a_{st})_{uv} =_{\text{def}} \begin{cases} a_{st} & \text{if } u = s \text{ and } v = t \\ a_{st}c_{tv} & \text{if } u = s \text{ and } v \neq t \\ c_{us}a_{st} & \text{if } u \neq s \text{ and } v = t \\ c_{us}a_{st}c_{tv} & \text{if } u \neq s \text{ and } v \neq t \end{cases}$$

Now entries in rows other than s may be non-zero, and the constants c_{uv} generalise $\top_{uv}$ used in the definition of h. Composition with these constants converts the type from $s \to t$ to $u \to v$ as required for an entry in row u and column v.

We show that this can work only with heavy restrictions placed on the constants c_{uv}. To this end, we look at the particular case in which the underlying typed 1-free omega algebra S is also a heterogeneous relation algebra [20]. The latter setting characterises total, univalent, surjective and injective relations as follows:

- R is total $\Leftrightarrow R\top = \top \Leftrightarrow 1 \leq RR^{\smile}$,
- R is univalent $\Leftrightarrow R^{\smile}R \leq 1$,
- R is surjective $\Leftrightarrow R^{\smile}$ is total,
- R is injective $\Leftrightarrow R^{\smile}$ is univalent,
- R is a bijective mapping $\Leftrightarrow R$ is total, univalent, surjective and injective.

Here $R^{\smile}$ is the converse of the relation R.

Theorem 10. *Assume that the domain S of $\hbar$ is a finitely typed 1-free omega algebra and a heterogeneous relation algebra. Assume that $\hbar$ preserves $\cdot$ and $\top$. Then all constants c_{uv} are bijective mappings.*

Proof. Let T be the finite pretype set of S. Let c_{uv} be one of the constants used to define $\hbar$. In particular, this implies that T contains two pretypes $u \neq v$. Then $c_{uv}\top_{vt} = \hbar(\top_{vt})_{ut} = \top_{ut}$ for every pretype $t \in T$ because $\hbar$ preserves $\top$. Hence c_{uv} is total, and this is equivalent to $1_{uu} \leq c_{uv}c_{uv}{}^{\smile}$. Moreover,

$$\begin{aligned} c_{vu}c_{uv} &= 1_{vv}c_{vu}c_{uv}1_{vv} = \hbar(1_{vv})_{vu}\hbar(1_{vv})_{uv} \leq \textstyle\sum_{x\in T} \hbar(1_{vv})_{vx}\hbar(1_{vv})_{xv} = (\hbar(1_{vv})\hbar(1_{vv}))_{vv} \\ &= \hbar(1_{vv}1_{vv})_{vv} = \hbar(1_{vv})_{vv} = 1_{vv} \end{aligned}$$

because $\hbar$ preserves the operation $\cdot$. Hence

$$c_{vu} = c_{vu}1_{uu} \leq c_{vu}c_{uv}c_{uv}{}^{\smile} \leq 1_{vv}c_{uv}{}^{\smile} = c_{uv}{}^{\smile} \ .$$

Symmetrically we obtain $c_{uv} \leq c_{vu}{}^{\smile}$, whence $c_{uv}{}^{\smile} \leq c_{vu}$. Together we get $c_{vu} = c_{uv}{}^{\smile}$. Therefore, $c_{uv}{}^{\smile}c_{uv} = c_{vu}c_{uv} \leq 1_{vv}$, which shows that c_{uv} is univalent. Symmetrically we obtain that c_{vu} is total and univalent, whence $c_{uv} = c_{vu}{}^{\smile}$ is surjective and injective. Thus c_{uv} is a bijective mapping. □

An example of a finitely typed 1-free omega algebra which is a heterogeneous relation algebra are the relations up to a fixed size, that is, the Boolean $m \times n$ matrices with $m, n \leq k$ for a fixed k. In that case, Theorem 10 implies that the elements are relations between sets of the same size, that is, all relations have a fixed, square size. But this is clearly too restrictive for heterogeneous relations, whence $\hbar$ cannot replace h.

5. Infinitely Typed Omega Algebras

The main result of this section extends Theorem 7 to infinitely typed 1-free omega algebras by additionally using ($\top$5). We then apply it to preserve the validity of infinitary formulas.

Theorem 11. *Every typed 1-free omega algebra satisfying* ($\top$1) *and* ($\top$5) *can be embedded into a 1-free omega algebra, except that the embedding need not preserve* $\top$. *Each of the axioms* ($\top$1)–($\top$5) *is preserved.*

Proof. Finitely typed algebras are covered by Theorem 7 using the mapping h. For infinitely typed algebras we also construct an embedding into a matrix algebra, but we have to be careful because infinite matrices do not directly support operations such as $\cdot$, $^+$ and $^\omega$.

Let $(S, +, \cdot, {}^+, {}^\omega, 0, \top)$ be a typed 1-free omega algebra with ($\top$1) and ($\top$5), based on an infinite set of pretypes T. As in [15], let a $T \times T$ matrix A be a matrix with a row s and a column t for each $s, t \in T$, such that the entry there (denoted by A_{st}) is an element of S_{st}, that is, $A_{st} : s \to t$. We cannot make the set of all $T \times T$ matrices a 1-free omega algebra, because the operation $\cdot$ on infinite matrices involves infinite sums which are not available in S. The plan therefore is to take an appropriate subset M of the matrices that can be represented finitely, define operations making M a 1-free omega algebra and embed S into M. An outline of the proof is as follows:

1. A subset M of the $T \times T$ matrices is defined.
2. Every infinite matrix in M has a finite representation.
3. Particular finite representations are fixed for every matrix in M.
4. The finite representations of the matrices in M form a 1-free omega algebra.
5. This 1-free omega algebra is lifted to M.
6. The operations on M do not depend on the choice of finite representations.
7. The typed 1-free omega algebra S is embedded into M.

We subsequently elaborate each of these steps.

1. We designate a subset M of the $T \times T$ matrices that will form the matrix algebra into which S is embedded. Every infinite matrix in M shall have a finite representation, so that we will be able to define the operations of the matrix algebra. Our finite representation compresses all but finitely many rows/columns into a single row/column. This is possible for all matrices in the subset M defined as follows.

Let M be the set of $T \times T$ matrices A for which there is a finite subset $T' \subseteq T$ such that, with relative complement $\overline{T'}$,

- $\forall s \in T' : \forall t, u \in \overline{T'} : A_{st} = A_{su} \top_{ut}$, that is, entries in row $s \in T'$ and columns $t, u \notin T'$ can be converted into each other by composition with $\top$,
- $\forall t \in T' : \forall s, u \in \overline{T'} : A_{st} = \top_{su} A_{ut}$, that is, entries in column $t \in T'$ and rows $s, u \notin T'$ can be converted into each other by composition to $\top$, and
- $\forall s, t, u, v \in \overline{T'} : A_{st} = \top_{su} A_{uv} \top_{vt}$, that is, entries in rows $s, u \notin T'$ and columns $t, v \notin T'$ can be converted into each other by composing $\top$ on both sides.

We denote the conjunction of these three properties as (F). The last property entails $\forall s, t, u \in \overline{T'} : A_{st} = A_{su} \top_{ut} = \top_{su} A_{ut}$ by ($\top$5). The choice of T' is not unique for a matrix A.

2. We give a finite representation of the matrices in M which is used to define the operations of 1-free omega algebra. The idea behind the representation is that due to the regularity caused by (F) just one pretype z from the infinite set $\overline{T'}$ suffices to reconstruct the entries for the other pretypes in $\overline{T'}$. Intuitively, the $T \times T$ matrix A is divided into four parts

$$\left(\begin{array}{c|c} A_{T'T'} & A_{T'z} \top_{z\overline{T'}} \\ \hline \top_{\overline{T'}z} A_{zT'} & \top_{\overline{T'}z} A_{zz} \top_{z\overline{T'}} \end{array} \right)$$

where the submatrix $A_{T'T'}$ is finite and square, and $z \in \overline{T'}$ is chosen arbitrarily. Then the rows/columns in $\overline{T'}$ are compressed into the single row/column z. Formally, we show that the infinite matrix A can be represented by the finite submatrix $A_{T'T'}$, the finite vector $A_{T'z}$, the finite transposed vector $A_{zT'}$ and the element A_{zz}. Together, they form the finite submatrix $A^{\downarrow} = A_{T''T''}$ with $T'' = T' \cup \{z\}$. Observe that $A^{\downarrow}$ satisfies (F), taking complements relative to T''. Conversely, for a $T'' \times T''$ matrix B that satisfies (F) with complements relative to T'', let $B^{\uparrow}$ be the $T \times T$ matrix given by

$$B^{\uparrow}{}_{st} = \begin{cases} B_{st} & \text{if } s,t \in T' \\ B_{sz}\top_{zt} & \text{if } s \in T' \text{ and } t \in \overline{T'} \\ \top_{sz}B_{zt} & \text{if } s \in \overline{T'} \text{ and } t \in T' \\ \top_{sz}B_{zz}\top_{zt} & \text{if } s,t \in \overline{T'} \end{cases}$$

Using ($\top$5) we get that $B^{\uparrow}$ satisfies (F):

- $\forall s \in T' : \forall t,u \in \overline{T'} : B^{\uparrow}{}_{st} = B_{sz}\top_{zt} = B_{sz}\top_{zu}\top_{ut} = B^{\uparrow}{}_{su}\top_{ut}$,
- $\forall t \in T' : \forall s,u \in \overline{T'} : B^{\uparrow}{}_{st} = \top_{sz}B_{zt} = \top_{su}\top_{uz}B_{zt} = \top_{su}B^{\uparrow}{}_{ut}$ and
- $\forall s,t,u,v \in \overline{T'} : B^{\uparrow}{}_{st} = \top_{sz}B_{zz}\top_{zt} = \top_{su}\top_{uz}B_{zz}\top_{zv}\top_{vt} = \top_{su}B^{\uparrow}{}_{uv}\top_{vt}$.

A few more calculations using (F) show $A^{\downarrow\uparrow} = A$ by

$$A^{\downarrow\uparrow}{}_{st} = \begin{cases} A^{\downarrow}{}_{st} = A_{st} & \text{if } s,t \in T' \\ A^{\downarrow}{}_{sz}\top_{zt} = A_{sz}\top_{zt} = A_{st} & \text{if } s \in T' \text{ and } t \in \overline{T'} \\ \top_{sz}A^{\downarrow}{}_{zt} = \top_{sz}A_{zt} = A_{st} & \text{if } s \in \overline{T'} \text{ and } t \in T' \\ \top_{sz}A^{\downarrow}{}_{zz}\top_{zt} = \top_{sz}A_{zz}\top_{zt} = A_{st} & \text{if } s,t \in \overline{T'} \end{cases}$$

and $B^{\uparrow\downarrow} = B$ if the same z is chosen:

$$B^{\uparrow\downarrow}{}_{st} = B^{\uparrow}{}_{st} = \begin{cases} B_{st} & \text{if } s,t \in T' \\ B_{sz}\top_{zt} = B_{st} & \text{if } s \in T' \text{ and } t \in \overline{T'} \text{ (hence } t = z) \\ \top_{sz}B_{zt} = B_{st} & \text{if } s \in \overline{T'} \text{ and } t \in T' \text{ (hence } s = z) \\ \top_{sz}B_{zz}\top_{zt} = B_{st} & \text{if } s,t \in \overline{T'} \text{ (hence } s = t = z) \end{cases}$$

Thus the matrix A is compressed to $A^{\downarrow}$, the compressed matrix B is uncompressed to $B^{\uparrow}$ and compression is lossless for matrices satisfying (F).

3. The argument above works for any $z \in \overline{T'}$, but we get a different representative $A^{\downarrow}$ for each choice. We now show how to fix the choice of z by including in S a copy of an arbitrary pretype. More precisely, take any $z \in T$ and construct a typed structure S' with pretypes $T''' = T \cup \{z'\}$ such that

$$\begin{array}{ll} S'_{st} = S_{st} & S'_{sz'} = S_{sz} \\ S'_{z't} = S_{zt} & S'_{z'z'} = S_{zz} \end{array}$$

for $s,t \in T$. The operations on S' are defined by calculating in S. There is a typed embedding of S' into S, which maps the pretype z' to z and is the identity otherwise. Hence S' is a typed 1-free omega algebra satisfying ($\top$1) and ($\top$5). Conversely, the identity is a typed embedding of S into S'. Each of the axioms ($\top$1)–($\top$5) is preserved. Thus we can use S' instead of S with the fixed pretype z' for the operations $^{\downarrow}$ and $^{\uparrow}$. We assume that this has been done in the beginning and continue writing S, T and z.

The argument above also works for any finite $T' \subseteq T$ such that (F) holds. In particular, if T' satisfies (F), so does any finite superset of T', giving a larger representative. We show below that the choice does not matter for the operations on M.

4. Our aim is to make M a 1-free omega algebra. The idea is to lift the operations from the finite representations to the infinite matrices. The finite matrices have the required structure as we show next.

The set of all $T'' \times T''$ matrices forms a 1-free omega algebra by Corollary 2. The set of $T'' \times T''$ matrices satisfying (F) forms a subalgebra M': it is closed under all operations. Namely, the finite 0- and $\top$-matrices satisfy (F) by ($\top$5). The operation $+$ on finite matrices preserves (F) by distributivity: for example,

$$(A+B)_{sz} = A_{sz} + B_{sz} = A_{sz}\top_{zz} + B_{sz}\top_{zz} = (A_{sz} + B_{sz})\top_{zz} = (A+B)_{sz}\top_{zz}$$

for $T'' \times T''$ matrices A, B satisfying (F). The remaining properties in (F) follow similarly. For $\cdot$ we also use distributivity: for example,

$$(A \cdot B)_{sz} = \sum_{u \in T''} A_{su}B_{uz} = \sum_{u \in T''} A_{su}B_{uz}\top_{zz} = (\sum_{u \in T''} A_{su}B_{uz})\top_{zz} = (A \cdot B)_{sz}\top_{zz}$$

only assuming that B satisfies (F). Again the remaining properties in (F) follow similarly. For $^+$, we unfold $A^+ = A + A \cdot A + A \cdot A^+ \cdot A$ and use the cases $+$ and $\cdot$. For $^\omega$, observe that the finite matrices satisfy ($\top$1) by Corollary 2, whence we unfold $A^\omega = A \cdot A^\omega \cdot \top$ with the $\top$-matrix, and apply the cases $\cdot$ and $\top$. It follows that M' is a 1-free omega algebra.

5. Making M a 1-free omega algebra, we define operations on M using the bijection to M' given by $^\downarrow$ and $^\uparrow$. For $A, B \in M$, let

$$\begin{array}{rclrclrcl}
A + B &=_{\text{def}}& (A^\downarrow + B^\downarrow)^\uparrow & \qquad A^+ &=_{\text{def}}& A^{\downarrow + \uparrow} & \qquad 0 &=_{\text{def}}& 0^\uparrow \\
A \cdot B &=_{\text{def}}& (A^\downarrow \cdot B^\downarrow)^\uparrow & \qquad A^\omega &=_{\text{def}}& A^{\downarrow \omega \uparrow} & \qquad \top &=_{\text{def}}& \top^\uparrow
\end{array}$$

For the binary operations, the same T'' is chosen for both $^\downarrow$ operations, namely, as the union of any individual choices for A and B. For the operations on M to be defined, we have to show that they do not depend on the choice of T''. This granted, an immediate consequence of the above definition is that $^\downarrow$ and $^\uparrow$ form an isomorphism between M and M'. Thus M is a 1-free omega algebra. Moreover, M satisfies any of ($\top$1)–($\top$5) if M' does so, and thus by Corollary 2 if S does so. It can be shown that $A + B$ is the componentwise sum and that $A \cdot B$ is the usual matrix product, where the involved infinite sums exist due to (F).

6. We show that different choices T_1 and T_2 for T'' lead to the same operations. Consider any finite T_1 and T_2 such that $T'' \subseteq T_1 \subseteq T_2 \subseteq T$. Assume without loss of generality that $T_2 = T_1 \cup \{y\}$ for some $y \in T \setminus T_1$; if the sets differ by more elements, the following argument can be applied several times. Denote by $^{\downarrow i}$ the restriction of $T \times T$ to $T_i \times T_i$ matrices, and by $^{\uparrow i}$ the respective inverse. We show that $^{\downarrow 1}$ and $^{\downarrow 2}$ induce the same operations.

- $0^{\uparrow 1} = 0^{\uparrow 2}$ since both generate the 0-matrix.
- $\top^{\uparrow 1} = \top^{\uparrow 2}$ since both generate the $\top$-matrix by ($\top$5).

Let $^{\downarrow 0} =_{\text{def}} {}^{\downarrow 1} \circ {}^{\uparrow 2}$ be the restriction of $T_2 \times T_2$ matrices to $T_1 \times T_1$ matrices. Then $^{\downarrow 1} = {}^{\downarrow 1} \circ {}^{\uparrow 2} \circ {}^{\downarrow 2} = {}^{\downarrow 0} \circ {}^{\downarrow 2}$. Moreover, $^{\uparrow 0} =_{\text{def}} {}^{\downarrow 2} \circ {}^{\uparrow 1}$ is the easily established inverse of $^{\downarrow 0}$. Hence $^{\uparrow 1} = {}^{\uparrow 2} \circ {}^{\downarrow 2} \circ {}^{\uparrow 1} = {}^{\uparrow 2} \circ {}^{\uparrow 0}$. For the sake of distinction, denote by $+_i$, $\cdot_i$, $^{+_i}$ and $^{\omega_i}$ the respective operations on $T_i \times T_i$ matrices. Then

- $A^{\downarrow 0} +_1 B^{\downarrow 0} = (A +_2 B)^{\downarrow 0}$ for any $T_2 \times T_2$ matrices A and B, since

  $$(A^{\downarrow 0} +_1 B^{\downarrow 0})_{st} = A^{\downarrow 0}{}_{st} + B^{\downarrow 0}{}_{st} = A_{st} + B_{st} = (A +_2 B)_{st} = (A +_2 B)^{\downarrow 0}{}_{st}$$

  for any $s, t \in T_1$. Thus

  $$(C^{\downarrow 1} +_1 D^{\downarrow 1})^{\uparrow 1} = (C^{\downarrow 2 \downarrow 0} +_1 D^{\downarrow 2 \downarrow 0})^{\uparrow 0 \uparrow 2} = (C^{\downarrow 2} +_2 D^{\downarrow 2})^{\downarrow 0 \uparrow 0 \uparrow 2} = (C^{\downarrow 2} +_2 D^{\downarrow 2})^{\uparrow 2}$$

  for any $T \times T$ matrices C and D.

- Using (F) with the fixed pretype z and the additional pretype y of T_2, we have $A_{sz}B_{zt} = A_{sy}\top_{yz}B_{zt} = A_{sy}B_{yt}$ for any $s,t \in T_1$ and $T_2 \times T_2$ matrices A and B. Therefore

$$(A^{\downarrow 0} \cdot_1 B^{\downarrow 0})_{st} = \sum_{u \in T_1} {A^{\downarrow 0}}_{su} {B^{\downarrow 0}}_{ut} = \sum_{u \in T_1} A_{su}B_{ut} = \sum_{u \in T_2} A_{su}B_{ut} = (A \cdot_2 B)_{st} = {(A \cdot_2 B)^{\downarrow 0}}_{st}$$

for any $s,t \in T_1$, from which the argument proceeds similarly to the case $+$.

- We show $G^{\downarrow 0^{+_1}} = G^{+_2\downarrow 0}$ for any $T_2 \times T_2$ matrix G, from which the claim follows since $E^{\downarrow 1^{+_1}\uparrow 1} = E^{\downarrow 2 \downarrow 0^{+_1}\uparrow 0 \uparrow 2} = E^{\downarrow 2^{+_2}\downarrow 0 \uparrow 0 \uparrow 2} = E^{\downarrow 2^{+_2}\uparrow 2}$ for any $T \times T$ matrix E. By (F) we assume

$$G = \begin{pmatrix} A & B & B\top_{zy} \\ C & D & D\top_{zy} \\ \top_{yz}C & \top_{yz}D & \top_{yz}D\top_{zy} \end{pmatrix}$$

for submatrix A, vector B, transposed vector C and element $D : z \to z$. Hence

$$G^{\downarrow 0^{+_1}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{+_1} \qquad \text{and} \qquad G^{+_2\downarrow 0} = \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} + \begin{pmatrix} B\top \\ D\top \end{pmatrix} (\top D \top)^* \begin{pmatrix} \top C & \top D \end{pmatrix} \right)^{+_1}$$

using the definition of $^+$ for matrices given in Section 2.4. Thus $G^{\downarrow 0^{+_1}} \leq G^{+_2\downarrow 0}$ since $^{+_1}$ is $\leq$-isotone. For the converse inequality, we have

$$\top_{zy}(\top_{yz}D\top_{zy})^*\top_{yz} = (\top_{zy}\top_{yz}D)^*\top_{zy}\top_{yz} = (\top_{zz}D)^*\top_{zz} = D^*\top_{zz}$$

by the sliding rule of typed 1-free Kleene algebra [14, 15], ($\top$5) and (F). Therefore

$$\begin{pmatrix} B\top \\ D\top \end{pmatrix} (\top D \top)^* \begin{pmatrix} \top C & \top D \end{pmatrix} = \begin{pmatrix} BD^*\top_{zz}C & BD^*\top_{zz}D \\ DD^*\top_{zz}C & DD^*\top_{zz}D \end{pmatrix} = \begin{pmatrix} BD^*C & BD^+ \\ D^+C & D^+D \end{pmatrix} \leq \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{+_1}$$

by (F). This implies $G^{+_2\downarrow 0} \leq G^{\downarrow 0^{+_1}}$ since $^{+_1}$ is increasing and idempotent.

- Along the lines of the case $^+$ it suffices to show $G^{\downarrow 0^{\omega_1}} = G^{\omega_2\downarrow 0}$ for any $T_2 \times T_2$ matrix G. In the following calculations, we use ($\top$1) and that $^{\omega_1}$ is $\leq$-isotone by ($\top$1). By the definition of $^\omega$ for matrices given in Section 2.4,

$$\begin{aligned} G^{\omega_2\downarrow 0} = {} & \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} + \begin{pmatrix} B\top \\ D\top \end{pmatrix} (\top D \top)^* \begin{pmatrix} \top C & \top D \end{pmatrix} \right)^{\omega_1} \begin{pmatrix} \top & \top \\ \top & \top \end{pmatrix} + {} \\ & \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{*_1} \begin{pmatrix} B\top \\ D\top \end{pmatrix} \left(\top D \top + \begin{pmatrix} \top C & \top D \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{*_1} \begin{pmatrix} B\top \\ D\top \end{pmatrix} \right)^{\omega_1} \begin{pmatrix} \top & \top \end{pmatrix} . \end{aligned}$$

The $T_1 \times T_1$ matrix with $\top$ entries vanishes by ($\top$1). Therefore

$$G^{\downarrow 0^{\omega_1}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\omega_1} \leq G^{\omega_2\downarrow 0} ,$$

whence it remains to show the converse inequality $G^{\omega_2\downarrow 0} \leq G^{\downarrow 0^{\omega_1}}$. First,

$$\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} + \begin{pmatrix} B\top \\ D\top \end{pmatrix} (\top D \top)^* \begin{pmatrix} \top C & \top D \end{pmatrix} \right)^{\omega_1} \leq \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{+_1\omega_1} = G^{\downarrow 0^{+_1\omega_1}} = G^{\downarrow 0^{\omega_1}}$$

as in the case $^+$. The last equality uses the property $a^{+^\omega} = a^\omega$ of 1-free omega algebra with ($\top$1), which follows since $a^{+^\omega} = a^+a^{+^\omega} = (a + aa^+)a^{+^\omega} = aa^{+^\omega}$ implies $a^{+^\omega} \leq a^\omega\top = a^\omega \leq a^{+^\omega}$. Second, because a similar calculation shows $a^*a^\omega = a^\omega$, it remains to show

$$\begin{pmatrix} B\top \\ D\top \end{pmatrix} \left(\top D \top + \begin{pmatrix} \top C & \top D \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{*_1} \begin{pmatrix} B\top \\ D\top \end{pmatrix} \right)^{\omega_1} \begin{pmatrix} \top & \top \end{pmatrix} \leq \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\omega_1} = G^{\downarrow 0^{\omega_1}} .$$

The parameter of ${}^{\omega_1}$ simplifies to $\top D^* C E^* B \top + \top F^* D \top$, where $E = A + BD^*C$ and $F = D + CA^*B$:

$$\begin{aligned}
&\top D \top + \begin{pmatrix} \top C & \top D \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{*_1} \begin{pmatrix} B\top \\ D\top \end{pmatrix} \\
&= \top D \top + \begin{pmatrix} \top C & \top D \end{pmatrix} \begin{pmatrix} E^+ & A^*BF^* \\ D^*CE^* & F^+ \end{pmatrix} \begin{pmatrix} B\top \\ D\top \end{pmatrix} + \begin{pmatrix} \top C & \top D \end{pmatrix} \begin{pmatrix} B\top \\ D\top \end{pmatrix} \\
&= \top D \top + \begin{pmatrix} \top C E^* + \top D D^* C E^* & \top C A^* B F^* + \top D F^* \end{pmatrix} \begin{pmatrix} B\top \\ D\top \end{pmatrix} \\
&= \top D \top + \begin{pmatrix} \top D^* C E^* & \top F F^* \end{pmatrix} \begin{pmatrix} B\top \\ D\top \end{pmatrix} = \top D \top + \top D^* C E^* B \top + \top F^+ D \top \\
&= \top D^* C E^* B \top + \top F^* D \top \ .
\end{aligned}$$

Now $EA^*BF^* = (A + BD^*C)A^*BF^* \leq AA^*BF^* + BF^*FF^* \leq A^*BF^*$ implies $E^*B \leq A^*BF^*$, whence

$$\top D^* C E^* B \top + \top F^* D \top \leq \top F^* C A^* B F^* \top + \top F^* F \top \leq \top F^+ \top \ .$$

It therefore suffices to show

$$\begin{pmatrix} B\top \\ F\top \end{pmatrix} (\top F^+ \top)^{\omega_1} \begin{pmatrix} \top & \top \end{pmatrix} \leq \begin{pmatrix} BF^\omega \top & BF^\omega \top \\ FF^\omega \top & FF^\omega \top \end{pmatrix} \leq \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\omega_1} ,$$

which reduces to $\top_{zy}(\top_{yz}F^+\top_{zy})^\omega \top_{yt} \leq F^{+\omega}\top_{zt} = F^\omega \top_{zt}$. By omega induction this follows from

$$\top_{zy}(\top_{yz}F^+\top_{zy})^\omega \top_{yt} = \top_{zy}\top_{yz}F^+\top_{zy}(\top_{yz}F^+\top_{zy})^\omega \top_{yt} = F^+\top_{zy}(\top_{yz}F^+\top_{zy})^\omega \top_{yt} \ ,$$

using $\top_{zy}\top_{yz}F^+ = \top_{zz}FF^* = FF^* = F^+$ due to ($\top$5) and (F).

This completes the independence of the operations on M of the choice of T''.

7. We finally embed S into M by the mapping $h'(a_{st}) = h(a_{st})^\uparrow$ using the embedding h of Theorem 7 from the subalgebra of S restricted to pretype set T'' into the algebra of $T'' \times T''$ matrices, where $T'' = \{s, t, u, z\}$ for any $u \in T$. Note that this uses a different embedding h for each S_{st}, but by ($\top$5) the matrix $h(a_{st})$ satisfies (F) and the result $h(a_{st})^\uparrow$ does not depend on the choice of u. The mapping h' is injective on each type and preserves the operations of 1-free omega algebra except $\top$ because h does so and ${}^\uparrow$ is an isomorphism from M' to M. The extra pretype u is needed to show that h' preserves $\cdot$ which involves three pretypes s, t, u. □

It follows that by additionally assuming ($\top$5), Corollaries 8 and 9 extend to infinitary formulas. Infinitary formulas are constructed from equations of expressions in the algebra by negation, possibly infinite conjunction, possibly infinite universal quantification and derived logical operations such as implication and possibly infinite disjunction. They may involve an infinite number of variables and their typed instances may involve an infinite number of types. Examples of infinitary formulas are the following ones:

- $\forall a, b, c, d : (\bigwedge_{i \in \mathbb{N}} ab^ic \leq ab^*c) \wedge ((\bigwedge_{i \in \mathbb{N}} ab^ic \leq d) \Rightarrow ab^*c \leq d)$ is *-continuity in Kleene algebra [6, 13].
- $a^\omega = 0 \Rightarrow \bigvee_{i \in \mathbb{N}} a^* \leq \sum_{0 \leq j \leq i} a^j$ states that a progressively finite element a is uniformly progressively bounded (there is a maximal length of the paths in a graph).
- $(\forall a_i)_{i \in \mathbb{N}} : \forall b : (\bigwedge_{i \in \mathbb{N}} a_i \leq b) \Rightarrow (\bigwedge_{i \in \mathbb{N}} \sum_{0 \leq j \leq i} a_j \leq b)$ is a simple semilattice law, the untyped form of which can be proved automatically in Isabelle using its Sledgehammer tool and integrated automated theorem provers.

Our results treat universal infinitary formulas. As usual, these are formulas in prenex form with only universal quantifiers.

Corollary 12. *A universal infinitary formula using only the operations* $+$, $\cdot$, ${}^{+}$, ${}^{\omega}$, 0 *is valid in 1-free omega algebra with* ($\top$1) *and* ($\top$5) *if and only if it is valid in typed 1-free omega algebra with* ($\top$1) *and* ($\top$5)*.*

Proof. The backward implication follows since every 1-free omega algebra is a typed 1-free omega algebra (with one type). We prove the forward implication.

Let S be a typed 1-free omega algebra with ($\top$1) and ($\top$5). Let H be the embedding of S into a 1-free omega algebra R with ($\top$1) and ($\top$5) according to Theorem 11. Let F be the given formula (without the prefix of universal quantifiers) holding in R. We show that every well-typed instance of F holds in S. To this end, let v be a valuation of its variables. In particular, $F(H(v))$ holds in R.

Consider an equation $a = b$ occurring in F, hence with expressions a and b using only the operations $+$, $\cdot$, ${}^{+}$, ${}^{\omega}$, 0. Since H is homomorphic, $a(H(v)) = b(H(v))$ is equivalent to $H(a(v)) = H(b(v))$. Moreover, the latter holds in R if and only if $a(v) = b(v)$ holds in S, because H is injective on the type of a and b.

Applying this equivalence to every equation in F, we obtain that $F(H(v))$ holds in R if and only if $F(v)$ holds in S. $\square$

Because the embeddings of Theorem 11 preserve ($\top$2), the same argument works for (typed) 1-free omega algebra with ($\top$1), ($\top$2), ($\top$5). We combine this with Corollary 4 extended to ($\top$1), ($\top$2), ($\top$5) and to infinitary formulas.

Corollary 13. *A universal infinitary formula using only the operations* $+$, $\cdot$, ${}^{+}$, ${}^{\omega}$, 0 *is valid in omega algebra if and only if it is valid in typed 1-free omega algebra with* ($\top$1), ($\top$2), ($\top$5)*.*

6. Typing Omega Algebras with $\top$: Positive Results

We extend the results of Section 5 to various kinds of formulas with $\top$. Throughout this section, H denotes either of two embeddings: the mapping h of Theorem 7 for finitely typed algebras and the mapping h' of Theorem 11 for infinitely typed algebras. Our results apply in both cases and we use H to state them uniformly and, whenever possible, to provide uniform proofs.

The results of Section 5 are restricted to formulas without $\top$ because $H(\top) = \top$ does not hold. We show, however, that H preserves $\top$ in many contexts: for example, $H(a\top) = H(a)\top$ holds. In other contexts, H preserves $\top$ if a particular mapping ρ_s is applied: for example, $\rho_s(H(\top a)) = \rho_s(\top H(a))$ where ρ_s is defined below. These weaker preservation properties suffice to extend our results to many formulas with $\top$ since H acts like an embedding on them.

We first introduce the mapping ρ_s mentioned above, stating a few consequences in Lemma 14, and then show the weaker preservation properties of H in Lemma 15. The subsequent discussion characterises the formulas for which these weaker properties suffice by looking at the initial symbols of the contained expressions. Moreover, aided by Lemma 16, such formulas can be reduced to a form which contains $\top$ at most in a few special places. This simplifies the proof of the key Theorem 17 that extends the weaker preservation properties of H to many expressions with $\top$. Corollary 18 shows that these weaker properties can be substituted into the construction of Corollary 12 to transfer the validity of a large class of untyped formulas with $\top$ to the typed setting.

We start with the mapping ρ_s that sets all entries of a matrix to 0 except those in row s:

$$(\rho_s(A))_{uv} =_{\mathrm{def}} \begin{cases} A_{uv} & \text{if } u = s \\ 0_{uv} & \text{if } u \neq s \end{cases}$$

Several consequences for ρ_s are stated in the following result.

Lemma 14. *Let A and B be matrices over a typed 1-free omega algebra with pretype set T. Then*

1. $\rho_s(AB) = \rho_s(A)B$*, provided AB exists,*
2. $\rho_s(A + B) = \rho_s(A) + \rho_s(B)$*,*
3. $\rho_s(A)^{\uparrow} = \rho_s(A^{\uparrow})$*, provided $s \in T'$ according to the definition of ${}^{\uparrow}$ in Theorem 11,*

4. $\rho_s(H(a_{st})) = H(a_{st})$.

PROOF.

1. The existence assumption is required for infinite matrices; otherwise the calculation is the same as for finite matrices. The entry in row u and column v of the matrix $\rho_s(AB)$ is

$$(\rho_s(AB))_{uv} = \left\{ \begin{array}{ll} (AB)_{uv} & \text{if } u = s \\ 0_{uv} & \text{if } u \neq s \end{array} \right\} = \left\{ \begin{array}{ll} \sum_{x\in T} A_{ux}B_{xv} & \text{if } u = s \\ 0_{uv} & \text{if } u \neq s \end{array} \right.$$

   The entry in row u and column v of the matrix $\rho_s(A)B$ is

$$(\rho_s(A)B)_{uv} = \sum_{x\in T} (\rho_s(A))_{ux}B_{xv} = \sum_{x\in T} \left\{ \begin{array}{ll} A_{ux}B_{xv} & \text{if } u = s \\ 0_{ux}B_{xv} & \text{if } u \neq s \end{array} \right\} = \left\{ \begin{array}{ll} \sum_{x\in T} A_{ux}B_{xv} & \text{if } u = s \\ 0_{uv} & \text{if } u \neq s \end{array} \right.$$

   since $0_{ux}B_{xv} = 0_{uv}$. Because the entries agree, $\rho_s(AB) = \rho_s(A)B$.
2. $\rho_s(A + B) = \rho_s(A) + \rho_s(B)$ follows immediately from the definition of ρ_s.
3. Let $s \in T'$. This implies that $\rho_s(A)$ satisfies property (F) of Theorem 11. Moreover, the entry in row u and column v of the matrix $\rho_s(A)^\uparrow$ is

$$\begin{aligned} (\rho_s(A)^\uparrow)_{uv} &= \left\{ \begin{array}{ll} (\rho_s(A))_{uv} & \text{if } u, v \in T' \\ (\rho_s(A))_{uz}\top_{zv} & \text{if } u \in T' \text{ and } v \in \overline{T'} \\ \top_{uz}(\rho_s(A))_{zv} & \text{if } u \in \overline{T'} \text{ and } v \in T' \\ \top_{uz}(\rho_s(A))_{zz}\top_{zv} & \text{if } u, v \in \overline{T'} \end{array} \right. \\ &= \left\{ \begin{array}{ll} A_{uv} & \text{if } u = s \text{ and } u, v \in T' \\ 0_{uv} & \text{if } u \neq s \text{ and } u, v \in T' \\ A_{uz}\top_{zv} & \text{if } u = s \text{ and } u \in T' \text{ and } v \in \overline{T'} \\ 0_{uz}\top_{zv} & \text{if } u \neq s \text{ and } u \in T' \text{ and } v \in \overline{T'} \\ \top_{uz}0_{zv} & \text{if } z \neq s \text{ and } u \in \overline{T'} \text{ and } v \in T' \\ \top_{uz}0_{zz}\top_{zv} & \text{if } z \neq s \text{ and } u, v \in \overline{T'} \end{array} \right\} = \left\{ \begin{array}{ll} A_{uv} & \text{if } u = s \text{ and } v \in T' \\ A_{uz}\top_{zv} & \text{if } u = s \text{ and } v \in \overline{T'} \\ 0_{uv} & \text{otherwise} \end{array} \right. \end{aligned}$$

   since $z \notin T' \Rightarrow z \neq s$ and $0_{uz}\top_{zv} = \top_{uz}0_{zv} = \top_{uz}0_{zz}\top_{zv} = 0_{uv}$ and $u = s \Rightarrow u \in T'$. The entry in row u and column v of the matrix $\rho_s(A^\uparrow)$ is

$$(\rho_s(A^\uparrow))_{uv} =_{\text{def}} \left\{ \begin{array}{ll} A^\uparrow{}_{uv} & \text{if } u = s \\ 0_{uv} & \text{if } u \neq s \end{array} \right\} = \left\{ \begin{array}{ll} A_{uv} & \text{if } u = s \text{ and } v \in T' \\ A_{uz}\top_{zv} & \text{if } u = s \text{ and } v \in \overline{T'} \\ 0_{uv} & \text{otherwise} \end{array} \right.$$

   again since $u = s \Rightarrow u \in T'$. Because the entries agree, $\rho_s(A)^\uparrow = \rho_s(A^\uparrow)$.
4. $\rho_s(h(a_{st})) = h(a_{st})$ holds because the entries of $h(a_{st})$ in rows other than s are 0. It follows that $\rho_s(h'(a_{st})) = \rho_s(h(a_{st})^\uparrow) = \rho_s(h(a_{st}))^\uparrow = h(a_{st})^\uparrow = h'(a_{st})$ by part 3. $\square$

Although the embedding H does not preserve $\top$ in general, part 1 of the following result shows it does in the context of a sequential composition, namely $H(a\top) = H(a)\top$. Moreover, part 3 shows that applying ρ_s helps to preserve $\top$ when composing from the other side.

Lemma 15. *Consider a typed 1-free omega algebra with* ($\top$5) *and let* $\top$ *be the* $\top$*-matrix. Then*

1. $H(a_{st}\top_{tu}) = H(a_{st})\top$,
2. $H(\top_{st}) = \rho_s(\top)$,
3. $H(\top_{st}a_{tu}) = \rho_s(\top H(a_{tu}))$,
4. $\top H(\top_{st}) = \top$.

PROOF. In parts 1–3, we first argue for typed algebras with a finite set of pretypes T, and then we conclude for infinitely typed algebras since $^\uparrow$ is an isomorphism as shown in Theorem 11 using ($\top$5).

1. The entry in row v and column w of the matrix $h(a_{st}\top_{tu})$ is

$$h(a_{st}\top_{tu})_{vw} = \left\{ \begin{array}{ll} a_{st}\top_{tu} & \text{if } v = s \text{ and } w = u \\ a_{st}\top_{tu}\top_{uw} & \text{if } v = s \text{ and } w \neq u \\ 0_{vw} & \text{if } v \neq s \end{array} \right\} = \left\{ \begin{array}{ll} a_{st}\top_{tw} & \text{if } v = s \\ 0_{vw} & \text{if } v \neq s \end{array} \right.$$

since $\top_{tu}\top_{uw} = \top_{tw}$ by ($\top$5). The entry in row v and column w of the matrix $h(a_{st})\top$ is

$$(h(a_{st})\top)_{vw} = \sum_{x\in T} h(a_{st})_{vx}\top_{xw} = \sum_{x\in T} \left\{ \begin{array}{ll} a_{st}\top_{xw} & \text{if } v = s \text{ and } x = t \\ a_{st}\top_{tx}\top_{xw} & \text{if } v = s \text{ and } x \neq t \\ 0_{vx}\top_{xw} & \text{if } v \neq s \end{array} \right\} = \left\{ \begin{array}{ll} a_{st}\top_{tw} & \text{if } v = s \\ 0_{vw} & \text{if } v \neq s \end{array} \right.$$

since $\top_{tx}\top_{xw} = \top_{tw}$ by ($\top$5) and $0_{vx}\top_{xw} = 0_{vw}$. Because the entries agree, $h(a_{st}\top_{tu}) = h(a_{st})\top$. It follows that $h'(a_{st}\top_{tu}) = h(a_{st}\top_{tu})^{\uparrow} = (h(a_{st})\top)^{\uparrow} = h(a_{st})^{\uparrow}\top^{\uparrow} = h'(a_{st})\top$.

2. The entries in row u and column v of the matrices $h(\top_{st})$ and $\rho_s(\top)$ agree because

$$h(\top_{st})_{uv} =_{\text{def}} \left\{ \begin{array}{ll} \top_{st} & \text{if } u = s \text{ and } v = t \\ \top_{st}\top_{tv} & \text{if } u = s \text{ and } v \neq t \\ 0_{uv} & \text{if } u \neq s \end{array} \right\} = \left\{ \begin{array}{ll} \top_{uv} & \text{if } u = s \\ 0_{uv} & \text{if } u \neq s \end{array} \right\} = (\rho_s(\top))_{uv}$$

since $\top_{ut}\top_{tv} = \top_{uv}$ by ($\top$5). Thus $h(\top_{st}) = \rho_s(\top)$. It follows that $h'(\top_{st}) = h(\top_{st})^{\uparrow} = \rho_s(\top)^{\uparrow} = \rho_s(\top^{\uparrow}) = \rho_s(\top)$ by Lemma 14.3.

3. $h(\top_{st}a_{tu}) = h(\top_{st})h(a_{tu}) = \rho_s(\top)h(a_{tu}) = \rho_s(\top h(a_{tu}))$ by Theorem 7, part 2 and Lemma 14.1. It follows that

$$h'(\top_{st}a_{tu}) = h(\top_{st}a_{tu})^{\uparrow} = \rho_s(\top h(a_{tu}))^{\uparrow} = \rho_s((\top h(a_{tu}))^{\uparrow}) = \rho_s(\top^{\uparrow}h(a_{tu})^{\uparrow}) = \rho_s(\top h'(a_{tu}))$$

by Lemma 14.3.

4. The entry in row u and column v of the matrix $\top h(\top_{st})$ is

$$\begin{aligned}(\top h(\top_{st}))_{uv} &= \sum_{x\in T} \top_{ux}h(\top_{st})_{xv} \\ &= \sum_{x\in T} \left\{ \begin{array}{ll} \top_{ux}\top_{xv} & \text{if } x = s \text{ and } v = t \\ \top_{ux}\top_{xt}\top_{tv} & \text{if } x = s \text{ and } v \neq t \\ \top_{ux}0_{xv} & \text{if } x \neq s \end{array} \right\} = \sum_{x\in T} \left\{ \begin{array}{ll} \top_{uv} & \text{if } x = s \\ 0_{uv} & \text{if } x \neq s \end{array} \right\} = \top_{uv}\end{aligned}$$

since $\top_{ux}\top_{xt}\top_{tv} = \top_{ux}\top_{xv} = \top_{uv}$ by ($\top$5) and $\top_{ux}0_{xv} = 0_{uv} \leq \top_{uv}$. Because the entries agree, $\top h(\top_{st}) = \top$. It follows that $\top h'(\top_{st}) = \top^{\uparrow}h(\top_{st})^{\uparrow} = (\top h(\top_{st}))^{\uparrow} = \top^{\uparrow} = \top$. $\square$

We therefore strive to eliminate occurrences of $\top$ in contexts other than $a\top$. The following facts are helpful for this task.

Lemma 16. *The following properties are valid in typed 1-free omega algebra with* ($\top$1), ($\top$2), ($\top$5)*:*

1. $0^+ = 0^\omega = 0$.
2. $\top^+ = \top^\omega = \top$.
3. $(\top a + b)^+ = \top a + \top ab^+ + b^+$.
4. $(\top a + b)^\omega = \top(a\top)^\omega + \top ab^\omega + b^\omega$.

Proof. Parts 3 and 4 apply Corollary 9 to derive typed instances of formulas valid in omega algebra.

1. $0^+_{ss} = 0_{ss} + 0_{ss}0^+_{ss} = 0_{ss} + 0_{ss} = 0_{ss}$ and $0^\omega_{ss} = 0_{ss}0^\omega_{ss} = 0_{ss}$.
2. $\top_{ss} \leq \top_{ss} + \top_{ss}\top^+_{ss} = \top^+_{ss} \leq \top_{ss}$ using ($\top$2).
   $\top_{ss} \leq \top_{ss}\top_{ss}$ by ($\top$5), whence $\top_{ss} \leq \top^\omega_{ss}\top_{ss} = \top^\omega_{ss} \leq \top_{ss}$ by omega induction, ($\top$1) and ($\top$2).

3. Observe that $c_{ss}^* \top_{st} = c_{ss}^+ \top_{st} + \top_{st} = \top_{st}$ for every c_{ss} by ($\top$2). Hence

$$\begin{aligned}(\top_{st} a_{ts} + b_{ss})^+ &= (b_{ss}^* \top_{st} a_{ts})^* b_{ss}^* (\top_{st} a_{ts} + b_{ss}) = (\top_{st} a_{ts})^* b_{ss}^* \top_{st} a_{ts} + (\top_{st} a_{ts})^* b_{ss}^+ \\ &= \top_{st} a_{ts} + (\top_{st} a_{ts})^* \top_{st} a_{ts} b_{ss}^+ + b_{ss}^+ = \top_{st} a_{ts} + \top_{st} a_{ts} b_{ss}^+ + b_{ss}^+\end{aligned}$$

using a typed instance of the decomposition property $(a+b)^+ = (a+b)^*(a+b) = (b^*a)^*b^*(a+b)$ which follows in Kleene algebra [14].
4. Again by the above observation,

$$\begin{aligned}(\top_{st} a_{ts} + b_{ss})^\omega &= (b_{ss}^* \top_{st} a_{ts})^\omega + (b_{ss}^* \top_{st} a_{ts})^* b_{ss}^\omega = (\top_{st} a_{ts})^\omega + (\top_{st} a_{ts})^* b_{ss}^\omega \\ &= (\top_{st} a_{ts})^\omega + (\top_{st} a_{ts})^* \top_{st} a_{ts} b_{ss}^\omega + b_{ss}^\omega = \top_{st} (a_{ts} \top_{st})^\omega + \top_{st} a_{ts} b_{ss}^\omega + b_{ss}^\omega\end{aligned}$$

using typed instances of the decomposition property $(a+b)^\omega = (b^*a)^\omega + (b^*a)^*b^\omega$ and the sliding property $(ba)^\omega = b(ab)^\omega$ known in omega algebra [5, 8]. □

It follows that these properties hold in 1-free omega algebra, too. We use them to apply the following transformations that *reduce* an expression of (typed) 1-free omega algebra:

- Eliminate all occurrences of 0 or transform the expression to 0 by using $0 + a = a + 0 = a$ and $0a = a0 = 0^+ = 0^\omega = 0$.
- Simplify occurrences of $\top$ in the context of $+$, $\cdot$, $^+$, $^\omega$ by using $\top + a = a + \top = \top\top = \top^+ = \top^\omega = \top$.
- Replace all products of sums with sums of products by using the distribution axioms $a(b+c) = ab+ac$ and $(a+b)c = ac+bc$.
- Lift any summand $\top a$ within the parameter of an operation $^+$ to the outside by using $(\top a + b)^+ = \top a + \top a b^+ + b^+$ and its instance $(\top a)^+ = \top a$.
- Lift any summand $\top a$ within the parameter of an operation $^\omega$ to the outside or replace it with $a\top$ by using $(\top a + b)^\omega = \top(a\top)^\omega + \top a b^\omega + b^\omega$ and its instance $(\top a)^\omega = \top(a\top)^\omega$. If $a = \top c$, first simplify by using $\top\top = \top$.

In a reduced expression, $\top$ occurs only in a context $a\top$, except perhaps in a summand $\top$ or $\top a$ of the outermost sum. The argument of Corollary 8 can be extended to reduced formulas without such summands. The set of initial symbols of an expression or equation of 1-free omega algebra, a concept well known in parser construction [21], is given by the recursively defined function

$$\begin{array}{lll} \mathit{first}(0) = \{0\} & \mathit{first}(e_1 e_2) = \mathit{first}(e_1) & \mathit{first}(e^+) = \mathit{first}(e) \\ \mathit{first}(\top) = \{\top\} & \mathit{first}(e_1 + e_2) = \mathit{first}(e_1) \cup \mathit{first}(e_2) & \mathit{first}(e^\omega) = \mathit{first}(e) \\ \mathit{first}(v) = \{v\} & \mathit{first}(e_1 = e_2) = \mathit{first}(e_1) \cup \mathit{first}(e_2) & \end{array}$$

where v is a variable and e, e_1, e_2 are expressions. The first two parts of the following result will help to distribute H over expressions in negative positions (such as the antecedent of an implication), while the third part will be applied to expressions in positive positions (such as the consequent of an implication).

Theorem 17. *Let S be a typed 1-free omega algebra with* ($\top$1), ($\top$2), ($\top$5). *Let e be an expression in S and let v be a valuation of its variables.*

1. *If $\top \notin \mathit{first}(e)$, then $H(e(v)) = e(H(v))$.*
2. *If $\mathit{first}(e) \subseteq \{0, \top\}$, then $\top H(e(v)) = e(H(v))$, where $\top$ is the $\top$-matrix.*
3. *$H(e(v)) = \rho_s(e(H(v)))$ for $e : s \to t$.*

Proof. Without loss of generality assume that e is reduced, since reduction according to the above transformations does not add $\top$ to $\mathit{first}(e)$ and maintains the property $\mathit{first}(e) \subseteq \{0, \top\}$. Throughout the proof we use that H is homomorphic as shown in Theorem 11.

1. Because $\top \notin \mathit{first}(e)$, the expression e does not have a summand $\top$ or $\top a$. Hence $\top$ occurs in e only in a context $a\top$. The claim follows by structural induction using Theorem 11 and Lemma 15.1, associating summands $a_1 a_2 a_3 \ldots a_{n-1} a_n$ as in $((\ldots((a_1 a_2) a_3) \ldots) a_{n-1}) a_n$.
2. Because $\mathit{first}(e) \subseteq \{0, \top\}$, either $e = \top$ or $e = \sum_i \top a_i$ with $\top \notin \mathit{first}(a_i)$. But $\top H(\top) = \top$ by Lemma 15.4 and

$$\top H(\textstyle\sum_i \top a_i(v)) = \top \sum_i H(\top a_i(v)) = \sum_i \top H(\top a_i(v)) = \sum_i \top H(\top) H(a_i(v)) = \sum_i \top a_i(H(v))$$

by Lemma 15.4 and part 1.
3. Either $e = \top$ or $e = \sum_i \top a_i + \sum_j b_j$ with $\top \notin \mathit{first}(a_i)$ and $\top \notin \mathit{first}(b_j)$. But $H(\top) = \rho_s(\top)$ by Lemma 15.2,

$$H(\textstyle\sum_i \top a_i(v)) = \sum_i H(\top a_i(v)) = \sum_i \rho_s(\top H(a_i(v))) = \rho_s(\sum_i \top H(a_i(v))) = \rho_s(\sum_i \top a_i(H(v)))$$

by Lemma 15.3, Lemma 14.2 and part 1, and

$$H(\textstyle\sum_j b_j(v)) = \sum_j H(b_j(v)) = \sum_j \rho_s(H(b_j(v))) = \rho_s(\sum_j H(b_j(v))) = \rho_s(\sum_j b_j(H(v)))$$

by Lemma 14.4, Lemma 14.2 and part 1. Hence also

$$H(\textstyle\sum_i \top a_i(v) + \sum_j b_j(v)) = \rho_s(\sum_i \top a_i(H(v)) + \sum_j b_j(H(v)))$$

by Lemma 14.2. □

The following result extends Corollary 12 to formulas whose negative equations contain $\top$ in a restricted way. An occurrence of an equation in a formula is positive/negative if it is in the scope of an even/odd number of negations (once implication and other derived operations are reduced to conjunction and negation).

Corollary 18. *A universal infinitary formula of 1-free omega algebra, in which every negative occurrence of an equation E satisfies $\top \notin \mathit{first}(E)$ or $\mathit{first}(E) \subseteq \{0, \top\}$, is valid in 1-free omega algebra with* ($\top$1), ($\top$2), ($\top$5) *if and only if it is valid in typed 1-free omega algebra with* ($\top$1), ($\top$2), ($\top$5).

Proof. The backward implication follows since every 1-free omega algebra is a typed 1-free omega algebra (with one type). We prove the forward implication.

Let S be a typed 1-free omega algebra with ($\top$1), ($\top$2), ($\top$5). Let H be the embedding of S into a 1-free omega algebra R with ($\top$1), ($\top$2), ($\top$5) according to Theorem 11. Let F be the given formula (without the prefix of universal quantifiers) holding in R. We show that every well-typed instance of F holds in S. To this end, let v be a valuation of its variables. In particular, $F(H(v))$ holds in R.

Consider the occurrence of an equation $a = b$ in F, where $a, b : s \to t$. For a positive occurrence, observe that $a(H(v)) = b(H(v))$ implies $H(a(v)) = \rho_s(a(H(v))) = \rho_s(b(H(v))) = H(b(v))$ by Theorem 17.3, and therefore $a(v) = b(v)$ since H is injective on the type $s \to t$. For a negative occurrence, observe that $a(v) = b(v)$ clearly implies $H(a(v)) = H(b(v))$ and $\top H(a(v)) = \top H(b(v))$, and therefore $a(H(v)) = b(H(v))$ by either Theorem 17.1 or Theorem 17.2.

Applying the respective implication to every positive/negative occurrence of an equation in F, we obtain a weakening of F, that is, if $F(H(v))$ holds in R then $F(v)$ holds in S. □

Because the embeddings of Theorem 11 preserve ($\top$3) and ($\top$4), the same argument works for (typed) 1-free omega algebra with ($\top$1)–($\top$5). We combine this with Corollary 6 extended to infinitary formulas.

Corollary 19. *A universal infinitary formula of 1-free omega algebra, in which every negative occurrence of an equation E satisfies $\top \notin \mathit{first}(E)$ or $\mathit{first}(E) \subseteq \{0, \top\}$, is valid in omega algebra if and only if it is valid in typed 1-free omega algebra with* ($\top$1)–($\top$5).

The above results extend to formulas whose expressions can be transformed using the axioms of (typed) 1-free omega algebra so as to satisfy the stated restrictions. An example of a formula which is not covered is $\top a = bc \Rightarrow \top ad = bcd$ because $\mathit{first}(\top a = bc) = \{\top, b\}$. Note that only the antecedent of an implication is restricted, not its consequent which may freely contain $\top$.

7. Conclusion

This paper shows that a large class of theorems of omega algebras can be transferred to typed omega algebras. It covers formulas which are universal, use $^+$ instead of *, do not contain 1 and satisfy a moderate restriction as regards the occurrence of $\top$ at the beginning of expressions in negated equations. Moreover, the formulas may be infinitary.

Typed algebras are used, for example, in automata constructions and matrix-based program representations. This paper brings existing untyped theorems, simpler proofs of new results and automated theorem proving to such applications.

Acknowledgement. I thank the anonymous referees for their valuable comments.

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