Towards a Typed Omega Algebra

Walter Guttmann

Department of Computer Science, University of Sheffield, UK
walter.guttmann@uni-ulm.de

Abstract. We propose axioms for 1-free omega algebra, typed 1-free omega algebra and typed omega algebra. They are based on Kozen's axioms for 1-free and typed Kleene algebra and Cohen's axioms for the omega operation. In contrast to Kleene algebra, several laws of omega algebra turn into independent axioms in the typed or 1-free variants.
We set up a matrix algebra over typed 1-free omega algebras by lifting the underlying structure. The algebra includes non-square matrices and care has to be taken to preserve type-correctness. The matrices can represent programs in total and general correctness. We apply the typed construction to derive the omega operation on two such representations, for which the untyped construction does not work.
We embed typed 1-free omega algebra into 1-free omega algebra, and this into omega algebra. Some of our embeddings, however, do not preserve the greatest element. We obtain that the validity of a universal formula using only $+$, $\cdot$, $^+$, $^\omega$ and 0 carries over from omega algebra to its typed variant. This corresponds to Kozen's result for typed Kleene algebra.

1 Introduction

Particular aspects of computations, such as non-termination, are conveniently treated by forming matrices over semirings [14]. A program is represented by a matrix as follows: some of the entries carry information about state transitions and non-terminating executions, while the remaining entries are specific constants. They are chosen and arranged so that matrix multiplication propagates the information as required to model sequential composition.

It is then possible to obtain the Kleene star and omega operations, which underlie the semantics of loops, by standard matrix constructions [3, 1, 10, 13]. Both operations can be derived for the matrices used in total correctness [7], and the Kleene star for the matrices used in general correctness [5]. The approach fails, however, for the omega operation in the latter case: the matrices used in general correctness are not closed under the construction given in [13].

In the present paper we solve this problem by typing the elements of the matrices. As regards the star operation, this means that the underlying structure is a typed Kleene algebra [11]. To deal with the omega operation, we propose a typed omega algebra, based on the untyped axiomatisation of [2].

Section 2 defines the necessary structures. Central to the present paper are (typed) 1-free omega algebras, an extension of the 1-free Kleene algebras of [11], which omit 1 and replace * by $^+$. While Kleene algebras are fairly similar to their 1-free variants, we identify several laws of omega algebra as independent axioms of 1-free omega algebra.

In Section 3 we show that finite matrices over typed 1-free omega algebras form (typed) 1-free omega algebras. To this end, we modify the matrix omega operation of [13] to obtain type-correct matrices. Particular subalgebras of the matrix algebra are then used to derive omega for the representation of programs. This works not only for general correctness but also for a recently introduced model that combines it with total correctness [8, 6].

In Section 4 we extend results of [11, 12], whereby restricted forms of universal statements are valid in the untyped setting if and only if they are valid in the typed setting. In particular, we embed typed 1-free omega algebra into 1-free omega algebra, and the latter into omega algebra. The embeddings require different subsets of axioms, and some do not preserve the greatest element.

Besides the application to program semantics, typed omega algebra can serve the following purposes. The ability to treat non-square matrices is useful for constructions related to automata [10], which indeed motivate typed Kleene algebra, and omega may be used to model infinite executions of the automata. Moreover, typed omega algebra is a subtheory of, and thus may yield insight into, heterogeneous relation algebra [16]; it fits into the hierarchy of [9].

2 Axioms

In this section we give axioms for (typed) (1-free) Kleene and omega algebras. Of these combinations, (typed or 1-free) omega algebras are new.

2.1 Omega Algebra

We recall the axioms of semirings, Kleene algebras and omega algebras. An idempotent semiring is a structure $(S, +, \cdot, 0, 1)$ that satisfies the following axioms:

$$\begin{array}{ccc} a + (b + c) = (a + b) + c & a(b + c) = ab + ac & a(bc) = (ab)c \\ a + b = b + a & (a + b)c = ac + bc & 1a = a \\ a + a = a & 0a = 0 & a1 = a \\ a + 0 = a & a0 = 0 & \end{array}$$

The operation $\cdot$ has higher precedence than $+$ and is frequently omitted by writing ab instead of $a \cdot b$. By $a \leq b \Leftrightarrow a + b = b$ we obtain the partial order $\leq$ on S with join $+$ and least element 0. The operations $+$ and $\cdot$ are $\leq$-isotone.

A Kleene algebra [10] is a structure $(S, +, \cdot, ^*, 0, 1)$ such that $(S, +, \cdot, 0, 1)$ is an idempotent semiring and the following axioms hold:

$$\begin{array}{cc} 1 + aa^* = a^* & b + ac \leq c \Rightarrow a^*b \leq c \\ 1 + a^*a = a^* & b + ca \leq c \Rightarrow ba^* \leq c \end{array}$$

The operation * is $\leq$-isotone and has highest precedence. Every Kleene algebra has the non-empty iteration $a^+ =_{\text{def}} aa^* = a^*a$. It satisfies $a^* = 1 + a^+$ and

$$a + aa^+ = a^+ \qquad b + ac \leq c \Rightarrow a^+b \leq c$$
$$a + a^+a = a^+ \qquad b + ca \leq c \Rightarrow ba^+ \leq c$$

The operation $^+$ is $\leq$-isotone and has the same precedence as *.

An omega algebra [2] is a structure $(S, +, \cdot, ^*, ^\omega, 0, 1)$ such that $(S, +, \cdot, ^*, 0, 1)$ is a Kleene algebra and the following axioms hold:

$$aa^\omega = a^\omega \qquad c \leq ac + b \Rightarrow c \leq a^\omega + a^*b$$

The operation $^\omega$ is $\leq$-isotone and has the same precedence as *. Every omega algebra has a $\leq$-greatest element $\top =_{\text{def}} 1^\omega$. It satisfies

$$a^\omega\top = a^\omega \qquad a \leq a\top \qquad \top = \top\top$$
$$a \leq \top \qquad a \leq \top a$$

We call those axioms of Kleene and omega algebra, which are implications, induction axioms.

2.2 1-Free Omega Algebra

We recall the axioms of 1-free Kleene algebras and introduce 1-free omega algebras. As discussed in Section 5, the restriction to 1-free algebras enables the transfer of universal formulas from the untyped to the typed setting.

A 1-free Kleene algebra [11] is a structure $(S, +, \cdot, ^+, 0)$ that satisfies the idempotent semiring axioms without 1, that is,

$$a + (b + c) = (a + b) + c \qquad a(b + c) = ab + ac \qquad a(bc) = (ab)c$$
$$a + b = b + a \qquad (a + b)c = ac + bc$$
$$a + a = a \qquad 0a = 0$$
$$a + 0 = a \qquad a0 = 0$$

and, replacing the *-axioms, the laws about $^+$ mentioned above:

$$a + aa^+ = a^+ \qquad b + ac \leq c \Rightarrow a^+b \leq c$$
$$a + a^+a = a^+ \qquad b + ca \leq c \Rightarrow ba^+ \leq c$$

An equivalent structure is obtained by replacing the implications with

$$ac \leq c \Rightarrow a^+c \leq c$$
$$ca \leq c \Rightarrow ca^+ \leq c$$

It follows that the operation $^+$ is $\leq$-isotone.

A 1-free omega algebra is a structure $(S, +, \cdot, ^+, ^\omega, 0, \top)$ such that $(S, +, \cdot, ^+, 0)$ is a 1-free Kleene algebra and the following axioms hold:

$$aa^\omega = a^\omega \qquad c \leq ac + b \Rightarrow c \leq a^\omega\top + a^+b + b$$

The operation $^\omega$ is not $\leq$-isotone in general, but $a \leq b$ implies both $a^\omega \leq b^\omega \top$ and $a^\omega \top \leq b^\omega \top$.

Observe the term $a^\omega \top$ replacing a^ω in the induction axiom to prepare it for the typed setting. We moreover consider the following axioms about $^\omega$ and $\top$:

($\top$1)	$a^\omega \top = a^\omega$	($\top$3)	$a \leq a\top$	($\top$5)	$\top = \top\top$
($\top$2)	$a \leq \top$	($\top$4)	$a \leq \top a$		

We explicitly state whenever they are used in addition to the axioms of 1-free omega algebra. Except for ($\top$5), which follows from ($\top$2) and either ($\top$3) or ($\top$4), these axioms are independent from each other and the axioms of 1-free omega algebra, as counterexamples generated by Mace4 witness.

To improve readability, we use the * notation also in 1-free algebras to abbreviate terms of the form

$$a^*b = a^+b + b \qquad ab^*c = ab^+c + ac$$
$$ba^* = ba^+ + b \qquad a^*bc^* = a^+bc^+ + a^+b + bc^+ + b$$

and similar ones, where * occurs in products with at least one 1-free element. For example, the omega induction axiom becomes $c \leq ac + b \Rightarrow c \leq a^\omega \top + a^*b$. Due to the semiring axioms, calculations using this notation work as expected. In such contexts * is $\leq$-isotone and the star induction axioms hold.

2.3 Typed 1-Free Omega Algebra

We use the mechanism for typing described in [11]. In particular, we assume a set T of pretypes $s, t, u, v, \ldots$ and obtain the set T^2 of types denoted as $s \to t$. The type of an expression a of omega algebra is denoted by $a : s \to t$ and can be derived using a type calculus with the rules

$$\frac{a, b : s \to t}{a + b : s \to t} \qquad \frac{a : s \to t \qquad b : t \to u}{ab : s \to u} \qquad \frac{a : s \to s}{a^*, a^+, a^\omega : s \to s} \qquad \begin{array}{c} 0, \top : s \to t \\ 1 : s \to s \end{array}$$

The rules for $^\omega$ and $\top$ are newly added. Expressions a and b in an equation $a = b$ must have the same type. We also write a_{st} to make clear that a has type $s \to t$.

For example, finite heterogeneous relations are modelled by letting T be the natural numbers. Then $a : s \to t$ denotes that a is a matrix with s rows and t columns. See [11] for further details about the typing mechanism.

A typed Kleene algebra (with pretypes T) is a set S of typed elements $a : s \to t$ ($s, t \in T$) with polymorphic operators $+$, $\cdot$, *, 0 and 1, typed according to the above inference rules, satisfying all well-typed instances of the Kleene algebra axioms.

Typed 1-free Kleene algebras and typed 1-free omega algebras are defined similarly, using all well-typed instances of the respective axioms in Section 2.2. All well-typed instances of a selection of ($\top$1)–($\top$5) may be considered besides.

For a typed omega algebra we use all well-typed instances of the omega algebra axioms, except for omega induction, which we replace by the omega induction axiom of 1-free omega algebra $c \leq ac + b \Rightarrow c \leq a^\omega \top + a^*b$.

A finitely typed algebra is one with finite T. We denote the set of elements with type $s \to t$ in a typed structure S by S_{st}. An untyped formula is valid in S if all its well-typed instances hold.

Remark. The axiom ($\top$2) establishes $\top : s \to t$ as the greatest element of type $s \to t$. As in heterogeneous relation algebra, each type has its own greatest element. In the untyped setting, being the greatest element is the main property of $\top$. In the typed setting, emphasis should be on its property to cause a change of types: from $a : s \to t$ we obtain the element $a\top$ of type $s \to u$ by multiplying with $\top : t \to u$. Thus ($\top$5) decomposes a type cast effected by $\top : s \to u$ into a sequence of two type casts effected by $\top : s \to t$ and $\top : t \to u$.

It is this type changing capacity which is used in the omega induction axiom. This ensures that $a^\omega\top$ is compatible with a^*b also if $b : s \to t$ with $s \neq t$. We have chosen to give a^ω the unique type $s \to s$ for $a : s \to s$, but another approach might give the more general type

$$\frac{a : s \to s}{a^\omega : s \to t}$$

which incorporates the type cast in a (more) polymorphic type of $^\omega$. While this would restore the original form of the omega induction axiom, in the 1-free case additional axioms are required to introduce $\top$. Even if 1 and * are available, it is not possible to derive all well-typed instances of ($\top$5): Sections 3.2, 3.3 and 5 feature models of typed omega algebra with ($\top$1)–($\top$4) but not ($\top$5).

3 Matrices

In this section we consider finite matrices over typed omega algebras. A matrix algebra is obtained by lifting the underlying structure. This is known for Kleene algebra [3, 10], typed Kleene algebra [11] and omega algebra [13]. For typed omega algebra, some modification is required to get the typing right.

The result shows how to obtain the omega operation for typed matrices. We apply it to matrix representations of programs in total and general correctness.

3.1 Matrices over typed 1-free omega algebra

Fix a typed 1-free omega algebra S with (not necessarily finite) pretypes T. We construct a typed 1-free omega algebra of finite matrices whose entries are elements of S. The pretypes of this matrix algebra are the finite sequences over T. Let $s_1, \dots, s_m \in T^m$ and $t_1, \dots, t_n \in T^n$ be pretypes, then a matrix has type $s_1, \dots, s_m \to t_1, \dots, t_n$ if and only if its size is $m \times n$ and, for each $1 \le i \le m$ and $1 \le j \le n$, the entry in row i and column j has type $s_i \to t_j$.

The operations $+$, $\cdot$, 0 and $\top$ are, as usual, the componentwise sum, the matrix product, the 0- and the $\top$-matrix, respectively. The non-empty iteration $^+$ is defined by $(a)^+ = (a^+)$ for 1×1 matrices and, inductively,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^+ = \begin{pmatrix} e^+ & a^*bf^* \\ d^*ce^* & f^+ \end{pmatrix} \qquad \text{with} \qquad \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} a + bd^*c \\ d + ca^*b \end{pmatrix} .$$

This is derived by $A^+ = AA^*$ from the usual matrix * of [3]. It is implicitly used in [11, Lemma 4.1], asserting that the resulting structure of square matrices satisfies the axioms of 1-free Kleene algebra. It is not difficult to check the axioms also for non-square matrices, hence we obtain a typed 1-free Kleene algebra.

The infinite iteration $^\omega$ is given by $(a)^\omega = (a^\omega)$ for size 1×1 and, inductively,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\omega = \begin{pmatrix} e^\omega & a^*bf^\omega \\ d^*ce^\omega & f^\omega \end{pmatrix} \begin{pmatrix} \top & \top \\ \top & \top \end{pmatrix} \qquad \text{with} \qquad \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} a + bd^*c \\ d + ca^*b \end{pmatrix} .$$

It is instructive to reflect on the type of the involved expressions. By its typing rule, the $^\omega$ operation is applied to a square matrix; its pretype is a finite sequence $t_1, \dots, t_n, t \in T^{n+1}$. For the inductive step, we take away the last element t of this sequence, denote by $s = t_1, \dots, t_n$ the remaining ones, and split the matrix accordingly into

$$\begin{pmatrix} a : s \to s & b : s \to t \\ c : t \to s & d : t \to t \end{pmatrix} .$$

Observe that e and f have the same types as a and d, respectively. Hence e^ω, a^*bf^ω, d^*ce^ω and f^ω have the types of a, b, c and d, respectively. The four $\top$ entries of the remaining matrix have these types as well:

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^\omega &= \begin{pmatrix} e^\omega & a^*bf^\omega \\ d^*ce^\omega & f^\omega \end{pmatrix} \begin{pmatrix} \top_{ss} & \top_{st} \\ \top_{ts} & \top_{tt} \end{pmatrix} \\ &= \begin{pmatrix} e^\omega \top_{ss} + a^*bf^\omega \top_{ts} & e^\omega \top_{st} + a^*bf^\omega \top_{tt} \\ f^\omega \top_{ts} + d^*ce^\omega \top_{ss} & f^\omega \top_{tt} + d^*ce^\omega \top_{st} \end{pmatrix} . \end{aligned}$$

Thus the resulting matrix has the correct type. Note that the columns of the matrix are not identical, as in the untyped case [13], but have their types adjusted. This is not necessary for the * and $^+$ operators.

It remains to show that the $^\omega$ operation thus defined satisfies the axioms. The proof is by induction on the size of the matrix, where the induction step assumes that the omega axioms hold for smaller matrices. The omega unfolding axiom is a consequence of

$$\begin{aligned} &\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e^\omega & a^*bf^\omega \\ d^*ce^\omega & f^\omega \end{pmatrix} = \begin{pmatrix} ae^\omega + bd^*ce^\omega & aa^*bf^\omega + bf^\omega \\ ce^\omega + dd^*ce^\omega & ca^*bf^\omega + df^\omega \end{pmatrix} \\ &= \begin{pmatrix} (a + bd^*c)e^\omega & (aa^*b + b)f^\omega \\ (c + dd^*c)e^\omega & (ca^*b + d)f^\omega \end{pmatrix} = \begin{pmatrix} ee^\omega & a^*bf^\omega \\ d^*ce^\omega & ff^\omega \end{pmatrix} = \begin{pmatrix} e^\omega & a^*bf^\omega \\ d^*ce^\omega & f^\omega \end{pmatrix} , \end{aligned}$$

using the induction hypothesis in the last step. For the omega induction axiom, let $u \in T$ and $x, p : s \to u$ and $y, q : t \to u$ such that

$$\begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} ax + by + p \\ cx + dy + q \end{pmatrix} ,$$

hence $y \leq d^\omega \top_{tu} + d^*(cx + q)$ by the induction hypothesis. Therefore

$$\begin{aligned} x &\leq ax + b(d^\omega \top_{tu} + d^*(cx + q)) + p = (a + bd^*c)x + b(d^\omega \top_{tu} + d^*q) + p \\ &\leq ex + b(f^\omega \top_{tu} + f^*q) + p , \end{aligned}$$

using $d \le f$ and $\le$-isotony. Once more by the induction hypothesis,

$$x \le e^\omega \top_{su} + e^*(bf^\omega \top_{tu} + bf^* q + p) \ .$$

We have to show

$$\begin{aligned}
\begin{pmatrix} x \\ y \end{pmatrix} &\le \begin{pmatrix} a & b \\ c & d \end{pmatrix}^\omega \begin{pmatrix} \top_{su} \\ \top_{tu} \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \begin{pmatrix} p \\ q \end{pmatrix} \\
&= \begin{pmatrix} e^\omega & a^*bf^\omega \\ d^*ce^\omega & f^\omega \end{pmatrix} \begin{pmatrix} \top_{ss} & \top_{st} \\ \top_{ts} & \top_{tt} \end{pmatrix} \begin{pmatrix} \top_{su} \\ \top_{tu} \end{pmatrix} + \begin{pmatrix} e^+ & a^*bf^* \\ d^*ce^* & f^+ \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix} \\
&= \begin{pmatrix} e^\omega(\top_{ss}\top_{su} + \top_{st}\top_{tu}) + a^*bf^\omega(\top_{ts}\top_{su} + \top_{tt}\top_{tu}) + e^*p + a^*bf^*q \\ f^\omega(\top_{ts}\top_{su} + \top_{tt}\top_{tu}) + d^*ce^\omega(\top_{ss}\top_{su} + \top_{st}\top_{tu}) + f^*q + d^*ce^*p \end{pmatrix} .
\end{aligned}$$

Consider x: by the above inequality and $e^\omega \le e^\omega \top_{ss}$ and $f^\omega \le f^\omega \top_{tt}$, it suffices to show $e^*bf^\omega \le a^*bf^\omega$ and $e^*bf^*q \le a^*bf^*q$. By star induction, these reduce to $ea^*bf^\omega \le a^*bf^\omega$ and $ea^*bf^*q \le a^*bf^*q$. Since $e = a + bd^*c$, these follow from $bd^*ca^*bf^\omega \le bf^\omega$ and $bd^*ca^*bf^*q \le bf^*q$. But these are consequences of $d^*ca^*b \le f^*f = f^+$ since $f^+f^\omega \le f^\omega$ and $f^+f^*q \le f^*q$. The inequality for y follows by swapping $\binom{x\ p\ a\ b\ e\ s}{y\ q\ d\ c\ f\ t}$.

The same argument applies column-wise to matrices having more than one column (replacing the vectors formed by x, y and p, q, respectively). This shows that the 1-free omega axioms hold for every well-typed instance. Moreover, the matrix algebra satisfies each of ($\top$1)–($\top$5) if the underlying typed 1-free omega algebra does so. We thus obtain the following result.

Theorem 1. *The finite matrices over a typed 1-free omega algebra form a typed 1-free omega algebra. Each of the axioms* ($\top$1)–($\top$5) *is preserved.*

It can be shown that the square matrices of size 2×2 and greater satisfy ($\top$1) vacuously. Only for the 1×1 matrices it is necessary that ($\top$1) holds in the underlying algebra.

For a given dimension n and sequence $(t_i) \in T^n$, the set of $n \times n$ matrices with type $(t_i) \to (t_i)$ is closed under the operations of 1-free omega algebra. We therefore obtain the following consequence of Theorem 1.

Corollary 2. *The $n \times n$ matrices with fixed type over a typed 1-free omega algebra form a 1-free omega algebra. Each of the axioms* ($\top$1)–($\top$5) *is preserved.*

By using diagonal matrices with 1-entries and the usual matrix * of [3], the above results also hold for typed omega algebras. Because the omega induction axioms of the typed and the untyped setting differ, the axiom ($\top$1) is needed for the version of Corollary 2 for typed omega algebras.

3.2 Matrices in General Correctness

We now apply the above theory to calculate the omega operation of so-called '(normal) prescriptions', which model programs in general correctness [4]. They are represented by matrices in [14, 5].

Let R be an omega algebra. A prescription is a 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R^{2\times 2}$$

such that $a = \top$ and $b = 0$ and $c = c\top$. Elements of the form $c\top$ model conditions; they are closed under the operations $+$ and $d\cdot$ for any $d \in R$.

The entry d records the terminating executions of a program, and c captures the set of states from which non-terminating executions exist. Fixing $a = \top$ and $b = 0$ ensures that the non-terminating executions of a program x become non-terminating executions of a sequential composition $x \cdot y$, but do not interfere with the terminating executions of $x \cdot y$, as fit for general correctness.

Trying to derive the omega operation on prescriptions using the untyped setting of [13], we obtain $e = \top + 0d^*c = \top$ and $f = d + c\top^*0 = d$, thus

$$\begin{pmatrix} \top & 0 \\ c & d \end{pmatrix}^\omega = \begin{pmatrix} e^\omega + \top^*0f^\omega & e^\omega + \top^*0f^\omega \\ f^\omega + d^*ce^\omega & f^\omega + d^*ce^\omega \end{pmatrix} = \begin{pmatrix} \top^\omega & \top^\omega \\ d^\omega + d^*c\top^\omega & d^\omega + d^*c\top^\omega \end{pmatrix}$$
$$= \begin{pmatrix} \top & \top \\ d^\omega + d^*c & d^\omega + d^*c \end{pmatrix} .$$

But this is not a prescription due to the entry $\top$ in the first row and the second column.

To solve this problem, let S' be the typed omega algebra with pretypes $T = \{1, 2\}$ such that $S'_{st} = R$ for each $s, t \in T$. The values of well-typed operations are given by calculating in R.

Now consider the substructure S of S' in which $S_{12} = \{0_{12}\}$ and $S_{st} = S'_{st}$ otherwise. Hence we restrict the type $1 \to 2$ to one element, retaining the other types. Then S is closed under the operations of typed omega algebra, except $\top$:

- The sum of two elements has type $1 \to 2$ only if both elements have this type, whence they are both 0_{12} and so is their sum.
- The product of two elements has type $1 \to 2$ only if one of them has this type, whence it is 0_{12} and so is the product.
- The operations *, $^\omega$, $^+$ and 1 do not apply to the type $1 \to 2$.

The constant 0_{12} is in S_{12} and we take $\top_{12} = 0_{12}$. The instance of the omega induction axiom $c \le ac + b \Rightarrow c \le a^\omega\top_{12} + a^*b$ holds since c must have the type $1 \to 2$, whence it is 0_{12}. The other axioms of typed omega algebra are satisfied since they hold in S' and S is closed. Therefore S is a typed omega algebra. Moreover, S satisfies ($\top$1)–($\top$4), but not ($\top$5) since $\top_{12}\top_{21} = 0_{12}\top_{21} = 0_{11} \ne \top_{11}$. Yet we have $\top_{11}^\omega\top_{11} = \top_{11}\top_{11} = \top_{11}$.

By Corollary 2 and ($\top$1), the 2×2 matrices over S form an omega algebra. But all prescriptions are elements of this matrix algebra, whence the omega operation is derived as shown in Section 3.1. Again, $e = \top_{11} + 0_{12}d^*c = \top_{11}$ and $f = d + c\top_{11}^*0_{12} = d$, thus

$$\begin{pmatrix} \top_{11} & 0_{12} \\ c & d \end{pmatrix}^\omega = \begin{pmatrix} e^\omega\top_{11} + \top_{11}^*0_{12}f^\omega\top_{21} & e^\omega\top_{12} + \top_{11}^*0_{12}f^\omega\top_{22} \\ f^\omega\top_{21} + d^*ce^\omega\top_{11} & f^\omega\top_{22} + d^*ce^\omega\top_{12} \end{pmatrix}$$
$$= \begin{pmatrix} \top_{11}^\omega\top_{11} & \top_{11}^\omega 0_{12} \\ d^\omega\top_{21} + d^*c\top_{11}^\omega\top_{11} & d^\omega\top_{22} + d^*c\top_{11}^\omega 0_{12} \end{pmatrix} = \begin{pmatrix} \top_{11} & 0_{12} \\ d^\omega\top_{21} + d^*c\top_{11} & d^\omega \end{pmatrix} .$$

In $R^{2\times 2}$ this simplifies to

$$\begin{pmatrix} \top & 0 \\ c & d \end{pmatrix}^\omega = \begin{pmatrix} \top & 0 \\ d^\omega\top + d^*c\top & d^\omega \end{pmatrix} = \begin{pmatrix} \top & 0 \\ d^\omega + d^*c & d^\omega \end{pmatrix},$$

since $d^\omega\top = d^\omega$ for $d \in R$ and $c\top = c$. The result is a prescription again.

Corollary 3. *Prescriptions are closed under the following operation* $^\omega$*, which satisfies the omega axioms:*

$$\begin{pmatrix} \top & 0 \\ c & d \end{pmatrix}^\omega = \begin{pmatrix} \top & 0 \\ d^\omega + d^*c & d^\omega \end{pmatrix}.$$

Note that the set of prescriptions is closed under * but does not form a subalgebra of the matrix algebra because the 0-matrix is not a prescription. Nevertheless, choosing different 0- and 1-elements, they form an omega algebra without right zero (the axiom $a0 = 0$ is omitted) [5].

3.3 Matrices in Total and General Correctness

We can also calculate the omega operation of so-called ‘extended designs’, which combine certain aspects of total and general correctness [8]. They too can be represented by matrices [6].

Let R be an omega algebra. An extended design is a 3×3 matrix of the form

$$\begin{pmatrix} \top & \top & \top \\ 0 & \top & 0 \\ p & q & r \end{pmatrix} \in R^{3\times 3}$$

such that $p\top = p \le q = q\top$ and $p \le r$.

Here, r records the terminating executions, q captures the non-terminating executions and p the aborting executions of a program. Again, the entries 0 and $\top$ are arranged to propagate this information according to the semantics of extended designs. The constraints $p \le q$ and $p \le r$ are typical for total correctness approaches: in the presence of an aborting execution, any other executions are considered irrelevant and hence absorbed.

Similarly to prescriptions, let S' be the typed omega algebra with pretypes $T = \{1, 2, 3\}$ such that $S'_{st} = R$ for each $s, t \in T$. Consider the substructure S of S' in which $S_{21} = S_{23} = \{0_{12}\}$ and $S_{st} = S'_{st}$ otherwise. Again, S is closed under the operations of typed omega algebra except $\top$. Closure under $\cdot$ is more involved now: for example, it would not suffice to collapse only the type $2 \to 3$, because an element c_{23} of this type may be obtained as the product $c_{23} = a_{21} \cdot b_{13}$. Taking $\top_{21} = 0_{21}$ and $\top_{23} = 0_{23}$, we again establish S as a typed omega algebra with ($\top$1)–($\top$4).

For the prescription submatrix of an extended design we obtain

$$\begin{pmatrix} \top_{22} & 0_{23} \\ q & r \end{pmatrix}^* = \begin{pmatrix} \top_{22} & 0_{23} \\ r^*q & r^* \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \top_{22} & 0_{23} \\ q & r \end{pmatrix}^\omega = \begin{pmatrix} \top_{22} & 0_{23} \\ r^\omega\top_{32} + r^*q & r^\omega \end{pmatrix}$$

as for Corollary 3. For the entire matrix we therefore obtain $e = \top_{11}$ and

$$\begin{aligned}
f &= \begin{pmatrix} \top_{22} & 0_{23} \\ q & r \end{pmatrix} + \begin{pmatrix} 0_{21} \\ p \end{pmatrix} \top_{11}^{*} \begin{pmatrix} \top_{12} & \top_{13} \end{pmatrix} = \begin{pmatrix} \top_{22} & 0_{23} \\ q & r \end{pmatrix} \\
d^{*}ce^{\omega} &= \begin{pmatrix} \top_{22} & 0_{23} \\ r^{*}q & r^{*} \end{pmatrix} \begin{pmatrix} 0_{21} \\ p \end{pmatrix} \top_{11}^{\omega} = \begin{pmatrix} \top_{22} & 0_{23} \\ r^{*}q & r^{*} \end{pmatrix} \begin{pmatrix} 0_{21} \\ p\top_{11} \end{pmatrix} = \begin{pmatrix} 0_{21} \\ r^{*}p \end{pmatrix} \\
f^{\omega} \begin{pmatrix} \top_{21} \\ \top_{31} \end{pmatrix} &= \begin{pmatrix} \top_{22} & 0_{23} \\ r^{\omega}\top_{32} + r^{*}q & r^{\omega} \end{pmatrix} \begin{pmatrix} 0_{21} \\ \top_{31} \end{pmatrix} = \begin{pmatrix} 0_{21} \\ r^{\omega}\top_{31} \end{pmatrix} \\
f^{\omega} \begin{pmatrix} \top_{22} & \top_{23} \\ \top_{32} & \top_{33} \end{pmatrix} &= \begin{pmatrix} \top_{22} & 0_{23} \\ r^{\omega}\top_{32} + r^{*}q & r^{\omega} \end{pmatrix} \begin{pmatrix} \top_{22} & 0_{23} \\ \top_{32} & \top_{33} \end{pmatrix} = \begin{pmatrix} \top_{22} & 0_{23} \\ r^{\omega}\top_{32} + r^{*}q & r^{\omega} \end{pmatrix}
\end{aligned}$$

and therefore

$$\begin{aligned}
\begin{pmatrix} \top & \top & \top \\ 0 & \top & 0 \\ p & q & r \end{pmatrix}^{\omega} &= \begin{pmatrix} e^{\omega}\top_{11} + a^{*}bf^{\omega} \begin{pmatrix} \top_{21} \\ \top_{31} \end{pmatrix} & e^{\omega} \begin{pmatrix} \top_{12} & \top_{13} \end{pmatrix} + a^{*}bf^{\omega} \begin{pmatrix} \top_{22} & \top_{23} \\ \top_{32} & \top_{33} \end{pmatrix} \\ f^{\omega} \begin{pmatrix} \top_{21} \\ \top_{31} \end{pmatrix} + d^{*}ce^{\omega}\top_{11} & f^{\omega} \begin{pmatrix} \top_{22} & \top_{23} \\ \top_{32} & \top_{33} \end{pmatrix} + d^{*}ce^{\omega} \begin{pmatrix} \top_{12} & \top_{13} \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} \top_{11} & \begin{pmatrix} \top_{12} & \top_{13} \end{pmatrix} \\ \begin{pmatrix} 0_{21} \\ r^{\omega}\top_{31} \end{pmatrix} + \begin{pmatrix} 0_{21} \\ r^{*}p \end{pmatrix} \top_{11} & \begin{pmatrix} \top_{22} & 0_{23} \\ r^{\omega}\top_{32} + r^{*}q & r^{\omega} \end{pmatrix} + \begin{pmatrix} 0_{21} \\ r^{*}p \end{pmatrix} \begin{pmatrix} \top_{12} & \top_{13} \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} \top_{11} & \top_{12} & \top_{13} \\ 0_{21} & \top_{22} & 0_{23} \\ r^{\omega}\top_{31} + r^{*}p & r^{\omega}\top_{32} + r^{*}q + r^{*}p\top_{12} & r^{\omega} + r^{*}p\top_{13} \end{pmatrix} .
\end{aligned}$$

In $R^{3\times 3}$ this further simplifies by $r^{\omega}\top = r^{\omega}$ and $r^{*}p\top = r^{*}p \leq r^{*}q$ to the following result. The argument is similar to that for prescriptions, using that extended designs are closed under * [6].

Corollary 4. *Extended designs are closed under the following operation* $^{\omega}$, *that satisfies the omega axioms:*

$$\begin{pmatrix} \top & \top & \top \\ 0 & \top & 0 \\ p & q & r \end{pmatrix}^{\omega} = \begin{pmatrix} \top & \top & \top \\ 0 & \top & 0 \\ r^{\omega} + r^{*}p & r^{\omega} + r^{*}q & r^{\omega} + r^{*}p \end{pmatrix} .$$

4 Typing

In this section we establish a class of theorems that can be transferred from the untyped to the typed setting. The typed setting thus profits from existing theorems, simpler (untyped) proofs of new theorems, and automated theorem provers (such as Prover9) which have no notion of types.

We proceed along the lines of [11] as far as possible. See [15] for a different approach (in Kleene algebra).

4.1 Embedding 1-Free Omega Algebra into Omega Algebra

It is proved in [11, Section 2.2] that every 1-free Kleene algebra can be embedded into a Kleene algebra. We extend that result to omega algebras.

Theorem 5. *Every 1-free omega algebra satisfying* $(\top 1)$ *and* $(\top 2)$ *can be embedded into an omega algebra, except that the embedding need not preserve* $\top$*.*

Proof. We extend the construction of [11]. Let S be a 1-free omega algebra, and construct the omega algebra $S' =_{\mathrm{def}} \{0,1\} \times S$ as follows. Intuitively, the element $(0,a)$ represents a, while $(1,a)$ represents $1+a$. The operations $+$, $\cdot$, *, 0 and 1 on S' are defined as in [11]:

$$\begin{aligned} (i,a)+(j,b) &= (i+j, a+b) \qquad & (i,a)^* &= (1,a^+) \qquad & 0 &= (0,0) \\ (i,a)\cdot(j,b) &= (ij, ab+ib+ja) & & & 1 &= (1,0) \end{aligned}$$

This uses $ix = x$ if $i = 1$, and $ix = 0$ if $i = 0$. We add the operation $^\omega$ by

$$(0,a)^\omega =_{\mathrm{def}} (0,a^\omega) \qquad \text{and} \qquad (1,a)^\omega =_{\mathrm{def}} (1,\top)\ .$$

The omega unfold axiom on S' follows since

$$\begin{aligned} (0,a)\cdot(0,a)^\omega &= (0,a)\cdot(0,a^\omega) = (0,aa^\omega) = (0,a^\omega) = (0,a)^\omega \\ (1,a)\cdot(1,a)^\omega &= (1,a)\cdot(1,\top) = (1,a\top+\top+a) = (1,\top) = (1,a)^\omega \end{aligned}$$

by $(\top 2)$. For the omega induction axiom $c' \le a'c' + b' \Rightarrow c' \le a'^\omega + a'^* b'$ of S' we consider two cases. If $a' = (1,a)$, then $c' \le (1,\top) = a'^\omega$ since $(1,\top)$ is the greatest element of S'. Otherwise, let $a' = (0,a)$ and $b' = (i,b)$ and $c' = (j,c)$. Then

$$(j,c) = c' \le a'c'+b' = (0,a)\cdot(j,c)+(i,b) = (0,ac+ja)+(i,b) = (i,ac+ja+b)\ .$$

Since the order on S' is componentwise, we have $j \le i$ and $c \le ac+ja+b$. Using the omega induction axiom of S, we obtain

$$c \le a^\omega\top + a^*(ja+b) = a^\omega + ja^+ + a^*b \le a^\omega + ia^+ + a^*b$$

by $(\top 1)$ and since $j \in \{0,1\}$. Therefore

$$\begin{aligned} c' = (j,c) &\le (i, a^\omega + ia^+ + a^*b) = (0,a^\omega) + (i, a^+b + b + ia^+) \\ &= (0,a)^\omega + (1,a^+)(i,b) = a'^\omega + (0,a)^*b' = a'^\omega + a'^*b'\ . \end{aligned}$$

The embedding $a \mapsto (0,a)$ is injective and a homomorphism, except that $\top$ is mapped to $(0,\top)$ which is not the greatest element $(1,\top)$ of S'. ☐

By using the embedding of Theorem 5 we obtain the following consequence about statements with universally quantified variables.

Corollary 6. *A universal formula using only the operators* $+$, $\cdot$, $^+$, $^\omega$, 0 *is valid in omega algebra if and only if it is valid in 1-free omega algebra with* $(\top 1)$ *and* $(\top 2)$.

If we admit further axioms, we can preserve $\top$ as well. The construction used in the following proof is not required for Kleene algebras.

Theorem 7. *Every 1-free omega algebra satisfying* ($\top$1)–($\top$4) *can be embedded into an omega algebra.*

Proof. We continue the proof of Theorem 5. Consider the smallest equivalence relation $\cong$ on S' which identifies $(0, \top) \cong (1, \top)$. It is a congruence:

- $(0, \top) + (i, a) = (i, \top + a) = (i, \top) \cong (1, \top) = (1, \top + a) = (1, \top) + (i, a)$ by ($\top$2). With commutativity we get congruence with respect to $+$.
- $(0, \top) \cdot (0, a) = (0, \top a) = (0, \top a + a) = (1, \top) \cdot (0, a)$ by ($\top$4). Moreover, $(0, \top) \cdot (1, a) = (0, \top a + \top) = (0, \top) \cong (1, \top) = (1, \top a + a + \top) = (1, \top) \cdot (1, a)$ by ($\top$2). Congruence in the second argument of $\cdot$ is analogous using ($\top$3).
- $(0, \top)^* = (1, \top^+) = (1, \top)^*$.
- $(0, \top)^\omega = (0, \top^\omega) = (0, \top) \cong (1, \top) = (1, \top)^\omega$, since $\top^\omega = \top$ holds in S: by ($\top$3) or ($\top$4) we have $\top \leq \top\top$, whence $\top \leq \top^\omega \top = \top^\omega \leq \top$ by omega induction, ($\top$1) and ($\top$2).

We thus obtain the embedding $a \mapsto [(0, a)]_\cong$ by composing the embedding of Theorem 5 with the canonical map h of $\cong$. Observe that h is injective on $S' \setminus \{(1, \top)\}$, thus the new embedding is injective as $(1, \top)$ is not in the image of the previous one. It remains to show that $S'/\cong$ is an omega algebra. The equational axioms follow since $S'/\cong$ is a homomorphic image of the omega algebra S'. We show the conditional equations:

- $ac \leq c \Rightarrow a^*c \leq c$: clear if $c = [(1, \top)]_\cong$ is the greatest element of $S'/\cong$. Otherwise, let $a', c' \in S'$ with $h(a') = a$ and $h(c') = c$ as h is surjective. Since h is a homomorphism, we have

$$h(a'c' + c') = h(a')h(c') + h(c') = ac + c = c = h(c') \ .$$

  Because c is not the greatest element, we have $c' \neq (1, \top) \neq a'c' + c'$, hence $a'c' + c' = c'$ since h is injective on $S' \setminus \{(1, \top)\}$. By star induction of S' we obtain $a'^*c' \leq c'$. Hence $a^*c = h(a')^*h(c') = h(a'^*c') \leq h(c') = c$ again since h is a homomorphism.
- $ca \leq c \Rightarrow ca^* \leq c$: symmetrically.
- $c \leq ac + b \Rightarrow c \leq a^\omega + a^*b$: if $ac + b \neq [(1, \top)]_\cong$, apply the previous argument. Otherwise, let $a', b', c' \in S'$ with $h(a') = a$ and $h(b') = b$ and $h(c') = c$. If $c' \leq a'c' + b'$, finish by applying omega induction of S' and the homomorphism h. Otherwise, $a'c' + b' = (0, \top)$ and $c' = (1, c'')$ for some $c'' \in S$. Hence $a' = (0, a'')$ for some $a'' \in S$. By ($\top$3),

$$a'c' \leq (0, a'')(1, \top) = (0, a''\top + a'') = (0, a''\top) = a'(0, \top) \ .$$

  Therefore $(0, \top) = a'c' + b' \leq a'(0, \top) + b'$, whence $(0, \top) \leq a'^\omega + a'^*b'$ by omega induction of S'. Thus $c \leq [(1, \top)]_\cong = h((0, \top)) \leq h(a'^\omega + a'^*b') = a^\omega + a^*b$ since h is a homomorphism. □

Corollary 8. *A universal formula of 1-free omega algebra is valid in omega algebra if and only if it is valid in 1-free omega algebra with* $(\top 1)$*–*$(\top 4)$*.*

Because $(\top 1)$–$(\top 4)$ are independent, these axioms are necessary for Theorem 7 and Corollary 8.

4.2 Embedding Typed 1-Free Omega Algebra into 1-Free Omega Algebra

It is proved in [11, Lemma 4.1] that every typed 1-free Kleene algebra can be embedded into a 1-free Kleene algebra. We extend that result to omega algebras.

As clarified in [12], a typed embedding is required to be injective for each type, but may map elements of distinct types to the same element.

The following result treats the case of finitely typed 1-free omega algebras. It can be generalised to infinitely typed 1-free omega algebras with $(\top 1)$ and $(\top 5)$, though that proof is more involved.

Theorem 9. *Every finitely typed 1-free omega algebra satisfying* $(\top 1)$ *can be embedded into a 1-free omega algebra, except that the embedding need not preserve* $\top$*. Each of the axioms* $(\top 1)$*–*$(\top 5)$ *is preserved.*

Proof. Let $(S, +, \cdot, {}^+, {}^\omega, 0, \top)$ be a typed 1-free omega algebra with $(\top 1)$, based on a set of n pretypes T. Arrange the pretypes in a fixed sequence $(t_i) \in T^n$. By Corollary 2, the $n \times n$ matrices with type $(t_i) \to (t_i)$ form a 1-free omega algebra, which satisfies any of $(\top 1)$–$(\top 5)$ if S does so.

We embed S into this matrix algebra by the mapping h defined as follows:

$$h(a_{st})_{uv} =_{\text{def}} \begin{cases} a_{st} & \text{if } u = s \text{ and } v = t \\ a_{st}\top_{tv} & \text{if } u = s \text{ and } v \neq t \\ 0_{uv} & \text{if } u \neq s \end{cases}$$

Thus the element $a : s \to t$ is mapped to a matrix with a in row s and column t, with $a\top$ in any other column of row s, and 0 in any other row. The embedding for 1-free Kleene algebra [11, Lemma 4.1] maps to 0 also in the second case.

Clearly h is injective on each type. We show that h preserves the operations of 1-free omega algebra except $\top$:

– Preservation of $+$ follows since $h(a_{st} + b_{st}) = h(a_{st}) + h(b_{st})$ by

$$\begin{aligned} h(a_{st} + b_{st})_{uv} &= \begin{cases} a_{st} + b_{st} & \text{if } u = s \text{ and } v = t \\ (a_{st} + b_{st})\top_{tv} = a_{st}\top_{tv} + b_{st}\top_{tv} & \text{if } u = s \text{ and } v \neq t \\ 0_{uv} = 0_{uv} + 0_{uv} & \text{if } u \neq s \end{cases} \\ &= h(a_{st})_{uv} + h(b_{st})_{uv} = (h(a_{st}) + h(b_{st}))_{uv} \ . \end{aligned}$$

– Preservation of $\cdot$ follows if we can show $h(a_{st}b_{tu})_{vw} = (h(a_{st})h(b_{tu}))_{vw}$. If $v \neq s$, then $h(a_{st}b_{tu})_{vw} = 0_{vw}$, but so is

$$(h(a_{st})h(b_{tu}))_{vw} = \sum_{x \in T} h(a_{st})_{vx} h(b_{tu})_{xw} = \sum_{x \in T} 0_{vx} h(b_{tu})_{xw} = 0_{vw} \ .$$

If $v = s$, then all the summands with $x \neq t$ vanish by $h(a_{st})_{vx} h(b_{tu})_{xw} = h(a_{st})_{vx} 0_{xw} = 0_{vw}$, hence

$$(h(a_{st})h(b_{tu}))_{vw} = \sum_{x \in T} h(a_{st})_{sx} h(b_{tu})_{xw} = h(a_{st})_{st} h(b_{tu})_{tw} = a_{st} h(b_{tu})_{tw} \ .$$

If $w = u$, this equals $a_{st} h(b_{tu})_{tu} = a_{st} b_{tu} = h(a_{st} b_{tu})_{vw}$, and if $w \neq u$, it equals $a_{st} b_{tu} \top_{uw} = h(a_{st} b_{tu})_{vw}$ as well.

– For a pretype $s \in T$, let $\top_{s\overline{s}}$ denote the transposed vector of all $\top_{st}$ elements such that $s \neq t \in T$, and similarly for the vector $0_{\overline{s}s}$ and matrix $0_{\overline{s}\,\overline{s}}$. Preservation of $^+$ follows by

$$\begin{aligned} h(a_{ss})^+ &= \begin{pmatrix} a_{ss} & a_{ss} \top_{s\overline{s}} \\ 0_{\overline{s}s} & 0_{\overline{s}\,\overline{s}} \end{pmatrix}^+ = \begin{pmatrix} a_{ss}^+ & a_{ss}^* a_{ss} \top_{s\overline{s}} 0_{\overline{s}\,\overline{s}}^* \\ 0_{\overline{s}\,\overline{s}}^* 0_{\overline{s}s} a_{ss}^* & 0_{\overline{s}\,\overline{s}}^+ \end{pmatrix} = \begin{pmatrix} a_{ss}^+ & a_{ss}^+ \top_{s\overline{s}} \\ 0_{\overline{s}s} & 0_{\overline{s}\,\overline{s}} \end{pmatrix} \\ &= h(a_{ss}^+) \ . \end{aligned}$$

– Preservation of $^\omega$ follows using ($\top$1) in

$$\begin{aligned} h(a_{ss})^\omega &= \begin{pmatrix} a_{ss} & a_{ss} \top_{s\overline{s}} \\ 0_{\overline{s}s} & 0_{\overline{s}\,\overline{s}} \end{pmatrix}^\omega = \begin{pmatrix} a_{ss}^\omega & a_{ss}^* a_{ss} \top_{s\overline{s}} 0_{\overline{s}\,\overline{s}}^\omega \\ 0_{\overline{s}\,\overline{s}}^* 0_{\overline{s}s} a_{ss}^\omega & 0_{\overline{s}\,\overline{s}}^\omega \end{pmatrix} \begin{pmatrix} \top_{ss} & \top_{s\overline{s}} \\ \top_{\overline{s}s} & \top_{\overline{s}\,\overline{s}} \end{pmatrix} \\ &= \begin{pmatrix} a_{ss}^\omega & 0_{s\overline{s}} \\ 0_{\overline{s}s} & 0_{\overline{s}\,\overline{s}} \end{pmatrix} \begin{pmatrix} \top_{ss} & \top_{s\overline{s}} \\ \top_{\overline{s}s} & \top_{\overline{s}\,\overline{s}} \end{pmatrix} = \begin{pmatrix} a_{ss}^\omega \top_{ss} & a_{ss}^\omega \top_{s\overline{s}} \\ 0_{\overline{s}s} & 0_{\overline{s}\,\overline{s}} \end{pmatrix} = \begin{pmatrix} a_{ss}^\omega & a_{ss}^\omega \top_{s\overline{s}} \\ 0_{\overline{s}s} & 0_{\overline{s}\,\overline{s}} \end{pmatrix} \\ &= h(a_{ss}^\omega) \ . \end{aligned}$$

– Clearly $h(0_{st})$ is the 0-matrix, but $h(\top_{st})$ is not the $\top$-matrix in general. $\square$

Already from this we obtain by modifying the argument of [11, Theorem 1.2] the following consequence. Formulas are implicitly assumed to be finitary.

Corollary 10. *A universal formula using only the operators* $+$, $\cdot$, $^+$, $^\omega$, 0 *is valid in 1-free omega algebra with* ($\top$1) *if and only if it is valid in typed 1-free omega algebra with* ($\top$1).

Proof. The backward implication follows since every 1-free omega algebra is a typed 1-free omega algebra (with one type). We prove the forward implication.

The given formula is equivalent to a conjunction of universal implications of the form $\bigwedge_{i \in I} a_i = b_i \Rightarrow \bigvee_{j \in J} c_j = d_j$ with finite index sets I and J and expressions a_i, b_i, c_j, d_j using only the operators $+$, $\cdot$, $^+$, $^\omega$, 0. We show the claim for such an implication F.

Assume F holds in 1-free omega algebra with ($\top$1). Let S be a typed 1-free omega algebra with ($\top$1) and pretypes T. Consider a well-typed instance F' of F. The instance F' only refers to finitely many pretypes $T' \subseteq T$. Let S' be the substructure of S restricted (in types, operations and axioms) to T'. The types which remain in S' keep all of their elements, whence all remaining axioms (equations and implications) still hold. In other words, S' is a finitely typed 1-free omega algebra with ($\top$1). Let h be the embedding of S' into a 1-free omega algebra R with ($\top$1) according to Theorem 9. In particular, F holds in R.

We show that F' holds in S'. To this end, let v be a valuation of its variables, and assume that the typed instance of each $a_i(v) = b_i(v)$ holds in S'. Then clearly $h(a_i(v)) = h(b_i(v))$ in R. Since h is homomorphic, $a_i(h(v)) = b_i(h(v))$ in R. By F we obtain $c_j(h(v)) = d_j(h(v))$ for some $j \in J$. Since h is homomorphic, $h(c_j(v)) = h(d_j(v))$ in R. Since h is injective on the type of c_j, we obtain $c_j(v) = d_j(v)$ in S'.

Every valuation of the variables of F' in S is a valuation in S', because it must respect the (fixed) types of the variables. Thus F' holds in S, too. □

Because the embedding of Theorem 9 preserves ($\top$2), the same argument works for 1-free omega algebra with ($\top$1) and ($\top$2). We combine this with Corollary 6.

Corollary 11. *A universal formula using only the operators* $+$, $\cdot$, $^+$, $^\omega$, 0 *is valid in omega algebra if and only if it is valid in typed 1-free omega algebra with* ($\top$1) *and* ($\top$2).

Whether the above results can be extended to formulas with $\top$ is open.

5 Conclusion

We conclude with remarks on the condition '1-free'. It is motivated by the counterexample $0 = 1 \Rightarrow a = b$ of [11], which holds in Kleene algebra, but not in typed Kleene algebra under its most general typing $0_{ss} = 1_{ss} \Rightarrow a_{tu} = b_{tu}$. It fails in those and only those typed Kleene algebras, where a type $s \to s$ is collapsed (has only one element) but another type $t \to u$ is not collapsed. An example is given by $T = \{1, 2\}$ and $S_{11} = S_{12} = S_{21} = \{0\}$ and $S_{22} = \{0, 1\}$ with operations defined as usual. In fact, the collapse of a type $s \to s$ triggers the collapse of all types $s \to t$ and $t \to s$: for example, $a_{ts} = a_{ts}1_{ss} = a_{ts}0_{ss} = 0_{ts}$. Further types, such as $t \to t$, are not affected. Sections 3.2 and 3.3 give applications with typed omega algebras where some but not all types are collapsed (though not a square type $s \to s$).

To avoid the above counterexample, further axioms have to be included: we propose ($\top$2) and ($\top$5). Assume a typed Kleene algebra with ($\top$2) and ($\top$5). Then $0_{ss} = 1_{ss}$ implies $\top_{ss} = \top_{ss}1_{ss} = \top_{ss}0_{ss} = 0_{ss}$. Moreover, $\top_{st} = 0_{st}$ for any $s, t \in T$ implies $\top_{uv} = \top_{us}\top_{sv} = \top_{us}\top_{st}\top_{tv} = \top_{us}0_{st}\top_{tv} = 0_{uv}$ by ($\top$5), and hence $a_{uv} = b_{uv}$ for any $a, b \in S_{uv}$ by ($\top$2). Including ($\top$2) and ($\top$5) propagates the collapse of one type to all types. For this reason, it is essential that the typed omega algebras in Sections 3.2 and 3.3 do not satisfy ($\top$5).

There are also models of typed omega algebra with ($\top$1)–($\top$4) but not ($\top$5), where none of the types is collapsed. An example is given by $T = \{1, 2\}$ where each type contains two Boolean 2×2 matrices under the usual matrix operations: $S_{11} = S_{22} = \{\emptyset, \{(0, 0), (1, 1)\}\}$ and $S_{12} = \{\emptyset, \{(0, 0)\}\}$ and $S_{21} = \{\emptyset, \{(1, 1)\}\}$.

On the other hand, we get another counterexample: $1_{ss} = \top_{ss} \Rightarrow 1_{tt} = \top_{tt}$. While the untyped implication clearly holds, the typed formula is not valid even in heterogeneous relation algebra [16], which is much more restricted than the typed omega algebras discussed in the present paper. Namely, $1 = \top$ holds for relations between one-element sets, but not for relations between larger sets.

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