On the inner structure of multirelations

Hitoshi Furusawa[a], Walter Guttmann[b,*], Georg Struth[c,d]

[a] *Department of Science, Kagoshima University, Japan*
[b] *Computer Science and Software Engineering, University of Canterbury, New Zealand*
[c] *Department of Computer Science, University of Sheffield, UK*
[d] *Collegium de Lyon, France*

Abstract

Binary multirelations form a model of alternating nondeterminism useful for analysing games, interactions of computing systems with their environments or abstract interpretations of probabilistic programs. We investigate this alternating structure with inner or demonic and outer or angelic choices in a relation-algebraic language extended with specific operations on multirelations that relate to the inner layer of alternation.

Keywords: Binary relations, Binary multirelations, Quantales, Semantics of computation

1. Introduction

This is the first article in a trilogy on the inner structure of multirelations, the determinisation of such relations [10] and their algebras of modal operators [11].

Multirelations – morphisms of type $X \nrightarrow \mathcal{P}Y$ in the category **Rel** – are established models of alternating nondeterminism. Elements (a, B), (a, C) of a multirelation can be interpreted as an outer nondeterministic or angelic choice between the subsets B or C of Y that depends on the element a of X, or as an outer nondeterministic evolution of a system from state a into the sets of states B or C. An element (a, B), in turn, can model the inner nondeterministic or demonic choices between the elements of B that depend on a, or an inner nondeterministic evolution from state a to any state in B. Multirelations have therefore been used as semantics for logics for games [2, 3, 25–27], for systems with alternating angelic/demonic nondeterminism [1, 7, 20], for systems with alternating forms of concurrency [28] or for abstract interpretations of probabilistic programs [22, 37, 38].

This article contributes to a line of work on algebras of multirelations [4, 5, 14, 15, 18] and algebraic languages for these [12], with specific operations for multirelations. A notable example of an operation on multirelations is their Peleg composition [28]: if $R : X \nrightarrow \mathcal{P}Y$ relates any a in X with a subset B of Y and if $S : Y \nrightarrow \mathcal{P}Z$ relates each $b \in B$ with a subset C_b of Z, then $R * S : X \nrightarrow \mathcal{P}Z$ relates a with the union of all the C_b. Detailed examples for the use of Peleg composition in computer science can be found, for instance, in [14, 28]. The operation is also crucial for the modal algebras on multirelations studied in the third part of this trilogy [11, 16, 23]. Its relationship with other compositions of multirelations has been studied in [12].

A typical operation on the inner or demonic structure is Peleg's parallel composition of multirelations [28]: if $R : X \nrightarrow \mathcal{P}Y$ and $S : X \nrightarrow \mathcal{P}Y$ relate any a in X with subsets B and C of Y, respectively, then $R \Cup S$ relates a with the inner or demonic choice $B \cup C$. We refer to this inner operation more neutrally as the inner union of R and S.

Further inner operations – an inner intersection, complementation and duality – have been considered by Rewitzky [29, 30]. An inner up-closure operation on multirelations – if R relates a with B and $B \subseteq C$, then R relates a with C – plays a key role in Parikh's game logic [25]. In

*Corresponding author, Email address: walter.guttmann@canterbury.ac.nz

an up-closed multirelation, each set of inner choices from any given element can be weakened to any superset with more inner choices. Rewitzky has added a dual down-closure operation, which supports strengthening inner choices to sets with fewer inner choices. She has also defined an inner preorder, akin to the Smyth preorder of domain theory, which relates R to S if the up-closure of R is contained in that of S and thus compares the inner nondeterminism of these multirelations.

Here, we add new results about the inner structure, the study of which was previously mainly targeted at games and up-closed multirelations. We investigate the closure of multirelations up-to preorder equivalence and relate them with quantales and Peleg composition, using tools and techniques from universal algebra. We also introduce a notion of convex closure, as the intersection of up- and down-closure, together with a corresponding preorder and equivalence, and study their properties. Up-closed multirelations are relevant to game logics. Down-closure is needed for characterising deterministic multirelations in the second article of this trilogy and modal operators on multirelations in the third article [10, 11]. In Theorem 3.2 we summarise the fact that homsets of multirelations form commutative quantales with either inner union or inner intersection as monoidal multiplication. These are isomorphic with respect to the duality induced by inner complementation, which replaces each set B in each pair (a, B) by its boolean complement. In Theorem 4.4 we prove that the down-closed and the up-closed elements in each homset form isomorphic subquantales of the double quantale on the entire homset, in which the inner intersection and the inner union collapses to (outer) intersection, respectively, and we show that the convex-closed elements also form quantales. In Theorem 5.5 we demonstrate that the quotient quantales on each homset with respect to the equivalences generated by the three preorders on multirelations are isomorphic to the quantales on up-, down- and convex-closed multirelations, respectively. In addition, we show in Section 5.3 that the inner preorders become partial orders, and even natural orders with a lattice structure, on certain subclasses of multirelations, and that they coincide on deterministic multirelations.

The identification of quantales in Theorems 3.2, 4.4 and 5.5 is particularly important for a more long-term research goal, namely the identification and axiomatisation of an algebra of multirelations similar to allegories [9] or relation algebras [36] extending the multirelational language in [12]. Yet this is difficult, as many multirelational concepts can be defined in different ways and the interactions between multirelational concepts can be quite complex. In fact, we can only take the initial steps in this trilogy of articles.

Nevertheless, with a view on such an axiomatisation, another main aim in this article is the exploration of the interplay of the different multirelational operations. Although we are working in concrete extensions and enrichments of **Rel**, we therefore focus on algebraic definitions of multirelational concepts, on proofs based on algebraic laws and on calculational properties. In this respect, we show in Proposition 3.9 that inner univalent and deterministic multirelations can be defined as fixpoints of functions definable in the multirelational language. An inner univalent multirelation is one where for each element (a, B) the inner set B contains at most one element. In Section 3.5 we algebraically reason about a simple game. In Proposition 4.6 we present a fixpoint characterisation for down-closed multirelations, using Peleg composition, and we show in Proposition 4.7 that Peleg composition preserves down-closure of multirelations, while similar properties for up-closed multirelations do not hold in general. In Theorem 5.16 we show that the inner properties studied in this article allow decomposing multirelations with respect to their outer and inner structure into inner deterministic parts. Further, in Section 6, we outline some results involving a dual to Peleg composition that interacts with inner intersection and related properties in the way Peleg composition interacts with inner union. Finally, we present many calculational properties that are either needed in proofs in our trilogy or might be helpful for shaping an axiomatisation of the algebra of multirelations in the future.

The concepts and results introduced in this article thus extend known, but dispersed notions and calculational results, and organise them in a systematic way that captures their algebraic structure precisely. This advances the understanding of the inner and outer structure of multirelations relative to previous work. The relevance of the properties developed here is evidenced in [10] and [11], where they support the study of deterministic multirelations using categories and power allegories, and that of modal algebras of multirelations, respectively.

The technical results in this trilogy of articles have benefitted greatly from working with the Isabelle/HOL proof assistant. In support of them we have developed a substantial library for multirelations [19], which extends a previous one [13] from single-homset multirelations to **Rel** and adds new results about the inner structure and beyond. While we have used this library to verify or falsify many conjectures related to this article and to increase our confidence in the correctness of our own definitions and proofs, we did not aim at a complete formalisation. This article is therefore self-contained without the Isabelle libraries, and not about formalised mathematics.

2. Relations and multirelations

We start with recalling the basics of binary relations and multirelations. See [12, 14–16, 28] for details. Our algebraic language of concrete relations and multirelations is based on enrichments of the category **Rel**, with sets as objects and binary relations as arrows. Among such enrichments are regular categories [17] and Dedekind categories [24], but our language is more closely related to relation-algebraic approaches [9, 33, 34], quantales [32] and their extensions with multirelational concepts [12]. We therefore start from concrete definitions in **Rel**, develop algebraic laws for them and then use algebraic reasoning as much as possible.

The relational calculus is rich and well documented [33, 34, 36]. Multirelations add a further layer of complexity which is much less explored. This richness sometimes prevents us from listing all properties used in calculations and proofs – we often refer to "standard" relational properties instead. We provide a dependency list of relational and multirelational concepts with respect to a small basis in Appendix A.

2.1. Binary relations

We consider binary relations as arrows in the category **Rel** and write $X \nrightarrow Y$ for the homset $\mathbf{Rel}(X,Y)$. The composition of arrows $R : X \nrightarrow Y$ and $S : Y \nrightarrow Z$ is relational composition $RS = \{ (a,b) \mid \exists c.\ R_{a,c} \wedge S_{c,b} \}$; identity arrows are relations $Id_X = \{ (a,a) \mid a \in X \}$. We compose arrows of categories in diagrammatic order, against the direction of function composition, but in the direction of relational composition. We often drop indices, writing Id for Id_X and likewise.

Each homset $\mathbf{Rel}(X,Y)$ forms a complete atomic boolean algebra, and relational composition preserves arbitrary sups in both arguments. We write $\emptyset_{X,Y}$ for the least and $U_{X,Y}$ for the greatest element in $X \nrightarrow Y$, $-R$ for the complement of R and $S - R$ for the relative complement $S \cap -R$.

The relation $R : X \nrightarrow X$ is a *test* if $R \subseteq Id$. Relational composition of tests is intersection. Tests form a subalgebra of $\mathbf{Rel}(X,X)$ for any X, a complete atomic boolean algebra.

We consider the following additional basic operations on relations:

- The *converse* of $R : X \nrightarrow Y$ is $R^{\smile} : Y \nrightarrow X, R \mapsto \{ (b,a) \mid R_{a,b} \}$.
- The *domain* of $R : X \nrightarrow Y$ is the test $dom(R) = \{ (a,a) \mid \exists b.\ R_{a,b} \}$ in $X \nrightarrow X$. It satisfies $dom(R) = Id_X \cap RR^{\smile} = Id_X \cap RU_{Y,X}$.
- The *left residual* of $T : X \nrightarrow Z$ and $S : Y \nrightarrow Z$ is given by
$$T/S = \bigcup \{ R : X \nrightarrow Y \mid RS \subseteq T \}.$$
- The *right residual* $T \backslash S : X \nrightarrow Y$ is given by $T \backslash S = (S^{\smile}/T^{\smile})^{\smile}$ for $T : Z \nrightarrow X$ and $S : Z \nrightarrow Y$.
- The *symmetric quotient* is $T \div S = (T \backslash S) \cap (T^{\smile}/S^{\smile})$.

By convention, unary operations have the highest precedence, followed by relational composition, followed by the lowest precedence for any other operations. Tests and domain elements form the same subalgebras. The residuals are right adjoints of relational composition.

We also need the following special relations:

- the *membership relation* $\in_Y : Y \nrightarrow \mathcal{P}Y$,
- the *subset relation* $\Omega_Y = \in_Y \backslash \in_Y = \{ (A, B) \mid A \subseteq B \subseteq Y \}$,
- the *complementation relation* $C = \in_Y \div -\in_Y = \{ (A, -A) \mid A \subseteq Y \}$.

We use the following properties of relations. Relation $R : X \nrightarrow Y$ is

- *total* if $dom(R) = Id_X$, or equivalently $Id_X \subseteq RR^{\smile}$,
- *univalent*, or a *partial function*, if $R^{\smile} R \subseteq Id_Y$,
- *deterministic*, or a *function*, if it is total and univalent.

Functions as deterministic relations in **Rel** are of course graphs of functions in **Set**.

The *power test* [12] $P_* : \mathcal{P}X \nrightarrow \mathcal{P}X$ of a test $P \subseteq Id_X$ is defined as

$$P_* = (\in_X \backslash P \in_X) \cap Id_{\mathcal{P}X} = \{(A, A) \mid \forall a \in A.\ (a, a) \in P\}.$$

It is needed in particular for the definition of the Peleg composition of multirelations in the following section.

Finally, we write $R|_A$ for the restriction of relation R to domain elements in the set A, $R(A)$ for the relational image of A under R and $R(a)$ for $R(\{a\})$.

Relations decompose into unions of partial functions. Each partial function contains one particular choice of codomain element (as a singleton set) for each domain element with a non-empty relational image. For $R, S : X \nrightarrow Y$, we write $S \subseteq_d R$ if S is univalent, $dom(S) = dom(R)$ and $S \subseteq R$.

Lemma 2.1. *Let $R : X \nrightarrow Y$. Then $R = \bigcup_{S \subseteq_d R} S$.*

2.2. Multirelations

A *multirelation* is an arrow $X \nrightarrow \mathcal{P}Y$ in **Rel**. We write $M(X, Y)$ for the homset $X \nrightarrow \mathcal{P}Y$.

Example 2.2. The $\in$-relation is a multirelation $X \nrightarrow \mathcal{P}X$. Graphs of nondeterministic functions $X \to \mathcal{P}Y$ are deterministic multirelations. An instance of this is $Id \div \in$, which relates every element to a singleton set containing it; see the units 1_X below.

Multirelations can be composed in many ways; see [12] for a comparison. The most relevant to us comes from concurrent dynamic logic [28].

The *Peleg composition* [28] $* : (X \nrightarrow \mathcal{P}Y) \times (Y \nrightarrow \mathcal{P}Z) \to (X \nrightarrow \mathcal{P}Z)$ can be defined in terms of the *Peleg lifting* $(-)_* : (X \nrightarrow \mathcal{P}Y) \to (\mathcal{P}X \nrightarrow \mathcal{P}Y)$ of multirelations [12]:

$$R * S = RS_* = \Big\{(a, C) \mid \exists B.\ R_{a,B} \wedge \exists f : Y \to \mathcal{P}Z.\ f|_B \subseteq S \wedge C = \bigcup f(B)\Big\},$$

where

$$R_* = \Big\{ (A, B) \mid \exists f : X \to \mathcal{P}Y.\ f|_A \subseteq R \wedge B = \bigcup f(A) \Big\}.$$

The Peleg lifting, in turn, satisfies $R_* = dom(R)_* \bigcup_{S \subseteq_d R} S_{\mathcal{P}}$, where the *Kleisli lifting* $(-)_{\mathcal{P}} : (X \nrightarrow \mathcal{P}Y) \to (\mathcal{P}X \nrightarrow \mathcal{P}Y)$ is given by

$$R_{\mathcal{P}} = \Big\{ (A, B) \mid B = \bigcup R(A) \Big\}.$$

Algebraically, $R_{\mathcal{P}} = \in R^{\smile} \in \div \in$.

In general, $R_* \neq \bigcup_{S \subseteq_d R} S_{\mathcal{P}}$ as the domain of the right-hand side contains sets with elements outside the domain of R. To omit these sets, we pre-compose with the power test $dom(R)_*$.

The units of Peleg composition are the multirelations

$$1_X = \{ (a, \{a\}) \mid a \in X \}.$$

Subsets of 1_X are multirelational tests. The Peleg lifting of such tests is the same as the power test of the corresponding relational test below Id_X. This is why we overload the notations for power test and Peleg lifting.

Peleg composition preserves arbitrary unions in its first argument, but only the order in its second one: $R \subseteq S \Rightarrow T * R \subseteq T * S$. Thus $\emptyset * R = \emptyset$, whereas the right zero law generally fails. It is not associative either; only $(R * S) * T \subseteq R * (S * T)$ holds. Hence multirelations do not form a category under Peleg composition. The composition becomes associative if the third factor is univalent or union-closed [12], a concept studied in Section 3.3.

3. Inner operations

The complete atomic boolean algebra of multirelations $X \nrightarrow \mathcal{P}Y$ forms an *outer* or *angelic* structure with *outer* operations and properties. In addition, the boolean algebra $\mathcal{P}Y$ on the second components of ordered pairs (a, A) forms a dual *inner* or *demonic* set structure for each a, with *inner* operations on multirelations. The parallel composition of concurrent dynamic logic [28], as discussed in the introduction, is an inner union operation; its algebraic properties are well studied [14, 15]. A dual inner intersection and an inner complementation that induces this duality have been defined by Rewitzky [29]. She refers to the inner operations as *power union*, *power intersection* and *power negation*. We now investigate the inner structure at greater detail and from a more structural point of view.

Recall that a *quantale* $(Q, \leq, \cdot, 1)$ is a complete lattice $(Q, \leq)$ and a monoid $(Q, \cdot, 1)$ such that $\cdot$ preserves all sups in both arguments, while a *quantale homomorphism* preserves all sups and the monoidal structure [31]. A quantale is *commutative* if $\cdot$ is.

Furthermore, we define the *natural order* $\leq$ for a semigroup $(S, \cdot)$ by $x \leq y \iff y = x \cdot y$ for all $x, y \in S$. Algebraic properties of $\cdot$ determine order properties: $\leq$ is transitive since $\cdot$ is associative; if $\cdot$ is commutative then $\leq$ is antisymmetric; $\cdot$ is idempotent if and only if $\leq$ is reflexive; 1 is a left unit of $\cdot$ if and only if 1 is the $\leq$-least element. If $\cdot$ is commutative and idempotent then $\cdot$ preserves the partial order $\leq$ in both arguments.

3.1. Definitions of inner operations

The *inner union*, *inner intersection*, their units and *inner complementation* are defined, for multirelations $R, S : X \nrightarrow \mathcal{P}Y$, as

$$R \Cup S = \{ (a, A \cup B) \mid R_{a,A} \wedge S_{a,B} \}, \qquad 1_{\Cup} = \{ (a, \emptyset) \mid a \in X \},$$
$$R \Cap S = \{ (a, A \cap B) \mid R_{a,A} \wedge S_{a,B} \}, \qquad 1_{\Cap} = \{ (a, Y) \mid a \in X \},$$
$$\sim R = \{ (a, -A) \mid R_{a,A} \}.$$

Algebraically, $\sim R = RC$, where C is the complementation relation from Section 2.1. Further, $1_{\Cap} = 1 \Cup \sim 1$ and $1_{\Cup} = 1 \Cap \sim 1$ for $1_{\Cap}, 1_{\Cup} : X \nrightarrow \mathcal{P}X$.

Remark 3.1. We do not know relation-algebraic definitions of $\Cup$ or $\Cap$ and conjecture that at least one of them, for instance $\Cup$, is necessary to obtain a basis for our multirelational language. See also Appendix A.

3.2. Algebra of inner operations

The interaction of $\Cup$ with $*$ and the outer operations is well known [14, 15]. The interactions of $\Cap$ follow by duality with respect to $\sim$. Before summarising these interactions in the next theorem, we define

$$M_{\Cup}(X, Y) = (M(X, Y), \subseteq, \Cup, 1_{\Cup}), \qquad M_{\Cap}(X, Y) = (M(X, Y), \subseteq, \Cap, 1_{\Cap}).$$

Theorem 3.2. *The structures $M_{\Cup}(X, Y)$ and $M_{\Cap}(X, Y)$ form commutative quantales. The inner complementation $\sim : M_{\Cup}(X, Y) \to M_{\Cap}(X, Y)$ is a quantale isomorphism (and its own inverse).*

Proof. The quantale structure of $M_{\Cup}(X,Y)$ has been checked in [15]; that of $M_{\Cap}(X,Y)$ follows from the isomorphism we establish next. First, $\sim$ is clearly involutive and surjective. Second, it is injective because $\sim R = \sim S$ implies $\sim\sim R = \sim\sim S$ and therefore $R = S$. Third, it is easy to verify that $\sim$ preserves the monoid structures of the quantales and arbitrary unions:

$$\sim(R \Cup S) = \sim R \Cap \sim S, \qquad \sim 1_{\Cup} = 1_{\Cap},$$
$$\sim(R \Cap S) = \sim R \Cup \sim S, \qquad \sim 1_{\Cap} = 1_{\Cup},$$
$$\sim\bigcup \mathcal{R} = \bigcup\{\, \sim R \mid R \in \mathcal{R}\,\}.$$
□

We call $\sim$ the *inner isomorphism* or *inner duality*, by contrast to the *outer isomorphism* or *outer duality* given by boolean complementation $-$. Properties of $\Cap$ thus translate from those of $\Cup$ via inner duality, and vice versa.

Remark 3.3. The quantales $M_{\Cup}(X,Y)$ and $M_{\Cap}(X,Y)$, as powerset structures, are boolean, atomic and completely distributive. The inner isomorphism preserves the boolean structure, $\sim -R = -\sim R$, as well as arbitrary intersections. In particular, $\sim\emptyset = \emptyset$ and $\sim U = U$, and zero laws $R \Cup \emptyset = \emptyset$ and $R \Cap \emptyset = \emptyset$ follow immediately from union preservation.

While $R \subseteq R \Cup R$, and dually $R \subseteq R \Cap R$, the operations $\Cup$ and $\Cap$ need not be idempotent and thus do not impose a semilattice structure on $M(X,Y)$. Thus neither $M_{\Cup}(X,Y)$ nor $M_{\Cap}(X,Y)$ forms a frame or locale, and the order $\subseteq$ of these powerset quantales is not the natural order on $\Cup$ or $\Cap$.

Example 3.4. The multirelation $R = \{(a,\{a\}),(a,\{b\})\}$ satisfies $R \Cup R = R \cup \{(a,\{a,b\})\}$ and $R \Cap R = R \cup \{(a,\emptyset)\}$.

Example 3.5. The greatest elements $U_{X,Y}$ in $M_{\Cup}(X,Y)$ or $M_{\Cap}(X,Y)$ are idempotents of $\Cup$ and $\Cap$. As in any semigroup, this induces subalgebras which have the $U_{X,Y}$ as units. The fixpoints of $(-) \Cup U$ are precisely the up-closed multirelations [15], which appear in the semantics of game logic [25]. By inner duality, the fixpoints of $(-) \Cap U$ yield down-closed multirelations.

The subalgebras arising from the idempotents U are studied in Section 4. Partial functions yield additional idempotents of the inner structure.

Lemma 3.6. *If $R : X \to \mathcal{P}Y$ is univalent, then $R \Cup R = R = R \Cap R$.*

Example 3.7. The converse does not hold: any $R = \{(a,A),(a,B)\}$ with $A \subset B$ is an idempotent for $\Cup$ and $\Cap$, but not univalent.

Remark 3.8. The relationship between $\Cup$ and $\Cap$ with $\subseteq$ differs from that of the outer operations. Implications between $R \subseteq S$, $R \Cup S = S$, $R \Cup S = R$, $R \Cap S = S$ and $R \Cap S = R$ can be refuted using small multirelations built from $(a,\emptyset)$, $(a,\{a\})$ and $\emptyset$. We obtain $(R \Cap S) \Cup T \subseteq (R \Cup T) \Cap (S \Cup T)$ and $(R \Cup S) \Cap T \subseteq (R \Cap T) \Cup (S \Cap T)$, but these properties do not imply order-preservation.

A *dual* operation $R^{\mathsf{d}} = -\sim R = -RC$ can be defined on multirelations [25, 29]. It combines inner and outer complementation. It is $\subseteq$-reversing and satisfies

$$\sim R = -R^{\mathsf{d}}, \qquad R^{\mathsf{d}^{\mathsf{d}}} = R, \qquad (R \cap S)^{\mathsf{d}} = R^{\mathsf{d}} \cup S^{\mathsf{d}}, \qquad (R \cup S)^{\mathsf{d}} = R^{\mathsf{d}} \cap S^{\mathsf{d}},$$
$$(-R)^{\mathsf{d}} = -(R^{\mathsf{d}}), \qquad (\sim R)^{\mathsf{d}} = \sim(R^{\mathsf{d}}).$$

3.3. Union-closure

Inner union and Peleg composition interact as follows [14]:

$$(R \Cup S) * T \subseteq (R * T) \Cup (S * T), \qquad R * (S \Cup T) \subseteq (R * S) \Cup (R * T),$$
$$T \Cup T \subseteq T \Rightarrow (R \Cup S) * T = (R * T) \Cup (S * T).$$

The distributivity law over inner unions in the first argument of Peleg composition generalises. We define

$$\Cup_{i\in I} R_i = \left\{ \left(a, \bigcup_{i\in I} A_i\right) \middle|\ \forall i \in I.\ (a, A_i) \in R_i \right\}$$

and call a multirelation R *union-closed* (or *additive* [29]) if $\Cup_{i\in I} R \subseteq R$ for all $I \neq \emptyset$, or equivalently, if $dom(S)(\in S^{\smile} \div \in) \subseteq R$ for all $S \subseteq R$ [12]. Then, for union-closed S,

$$(\Cup_{i\in I} R_i) * S = \Cup_{i\in I} (R_i * S).$$

On the other hand, $(R * S) \Cup (R * T) \nsubseteq R * (S \Cup T)$ even for union-closed R, S and T [15]. Note that $R \Cup R \subseteq R$ and therefore $R \Cup R = R$ if R is union-closed, but this is not an equivalence: union-closure includes arbitrary unions of target sets, not just finite unions.

Union-closed multirelations arise in the study of probabilistic systems [22, 37, 38]. In this context, probability distributions are assumed to satisfy a convexity condition [21]. An abstraction of probabilistic systems to multirelations is proposed in [22, 38], and this abstraction translates the convexity condition to union-closure.

3.4. Inner determinism, inner univalence

The inner structure of multirelations leads to notions of inner univalence, inner totality and inner determinism, which we study in this section. Inner total multirelations have been called *total*, outer total multirelations *proper*, inner univalent multirelations *angelic* and outer univalent multirelations *demonic* in [29, 30]. Here we introduce multirelational atoms to define these notions algebraically. We then characterise inner univalent, total and deterministic multirelations as fixpoints of functions that can be expressed in our multirelational language. We conclude this section with some structural properties.

The relation $U_{X,Y}$ is mapped by $(-)1_Y$ to

$$\mathsf{A}_{\Cup} = U_{X,Y} 1_Y = \{ (a, \{b\}) \mid a \in X \wedge b \in Y \},$$

the set of all (multirelational) *atoms* in $M(X, Y)$. By inner duality,

$$\mathsf{A}_{\Cap} = \{ (a, Y - \{b\}) \mid a \in X, b \in Y \}$$

is the set of all *co-atoms*. Of course, $\sim\mathsf{A}_{\Cup} = \mathsf{A}_{\Cap}$ and $\sim\mathsf{A}_{\Cap} = \mathsf{A}_{\Cup}$. Atoms allow expressing inner analogues to (outer) determinism, univalence and totality.

The multirelation $R : X \nrightarrow Y$ is

- *inner univalent* if $R \subseteq \mathsf{A}_{\Cup} \cup 1_{\Cup}$, that is, B is either a singleton or empty for each $(a, B) \in R$,
- *inner total* if $R \subseteq -1_{\Cup}$, that is, B is non-empty for each $(a, B) \in R$,
- *inner deterministic* if it is inner univalent and inner total, that is, $B \subseteq Y$ is a singleton set for each $(a, B) \in R$.

Inner deterministic multirelations are obviously subsets of $\mathsf{A}_{\Cup}$.

Inner univalent multirelations thus admit only outer or angelic choices, but not inner ones; they are completely angelic. Outer univalent multirelations, by contrast, admit only inner or demonic choices, but not outer ones; they are completely demonic. Inner deterministic multirelations can therefore be seen as strictly angelic, as all inner choices must be non-empty, and outer deterministic multirelations as strictly demonic, as empty outer choices are impossible.

Next we characterise the inner univalent, total and deterministic multirelations as fixpoints.

Proposition 3.9.

1. *The inner univalent multirelations are the fixpoints of* $(-) \cap (\mathsf{A}_{\Cup} \cup 1_{\Cup})$.
2. *The inner total multirelations are the fixpoints of* $(-) - 1_{\Cup}$.

3. *The maps* $(-) \cap \mathsf{A}_{\Cup}$ *and* $(-)1^{\smile}1$ *coincide. The fixpoints of these maps are the inner deterministic multirelations.*

Proof. We only prove that $R \cap \mathsf{A}_{\Cup} = R1^{\smile}1$ for any multirelation R. This follows immediately from the relational law

$$PQ \cap S = (P \cap SQ^{\smile})Q$$

for outer univalent Q [34], instantiated with $Q = 1$:

$$\mathsf{A}_{\Cup} \cap R = U1 \cap R = (U \cap R1^{\smile})1 = R1^{\smile}1. \qquad \square$$

The Peleg composition of inner deterministic multirelations becomes much simpler.

Lemma 3.10. *Let R, S, T be multirelations of appropriate types and R inner deterministic. Then*

1. $R * S = R1^{\smile}S$,
2. $R * (S * T) = (R * S) * T$,

Proof. For (1), $R * S = R1^{\smile}1S_* = R1^{\smile}(1 * S) = R1^{\smile}S$, using Proposition 3.9 in the first step. Then (2) follows from (1) because

$$R * (S * T) = R1^{\smile}(S * T) = R1^{\smile}ST_* = (R1^{\smile}S) * T = (R * S) * T. \qquad \square$$

Remark 3.11. Lemma 3.10 is used for relating down-closed multirelations with Peleg composition in Section 4.3. Item (2) in this lemma raises the question whether inner deterministic multirelations under Peleg composition form a category. The answer is positive, as we show in [10], where inner and outer determinism and univalence are studied more systematically.

Finally, we mention the following preservation laws for univalence, totality and determinism – inner and outer – without proof (a formal verification can be found in our Isabelle theories).

Lemma 3.12. *Inner unions preserve outer univalence, inner and outer totality, and outer determinism; inner intersections preserve inner and outer univalence, outer totality and outer determinism.*

3.5. Case study: Nim

We apply the above concepts and results in the study of a variant of the Nim game. Our formalisation is similar to that of [20], except that we use Peleg composition with union-closed multirelations instead of Parikh composition with up-closed ones. Peleg composition with union-closed multirelations can be seen as an abstraction of a probabilistic game: the first player makes a probabilistic (inner) choice, the second an angelic (outer) choice. See [22, 37] for further details about this abstraction of probabilistic systems to multirelations.

Recall that in the Nim game two players alternate in removing a number of matches from a given pile. Let the multirelation $S_i = \{(x, \{x-i\}) \mid x \in \mathbb{Z}\} : \mathbb{Z} \nrightarrow \mathcal{P}\mathbb{Z}$ describe the action of taking i matches from the pile. It is inner and outer deterministic and union-closed. The first player is modelled by the inner choice $P = S_1 \Cup S_2$, the second player by the outer choice $Q = S_1 \cup S_2 \cup P$ (taking the union with P makes Q union-closed). Then

$P * Q$

[Section 3.3: $*$ with union-closed Q distributes over $\Cup$ in P]

$= (S_1 * Q) \Cup (S_2 * Q)$

[[10, Proof of Lemma 5.9]: $*$ of inner univalent S_i distributes over $\cup$ in Q]

$= ((S_1 * S_1) \cup (S_1 * S_2) \cup (S_1 * P)) \Cup ((S_2 * S_1) \cup (S_2 * S_2) \cup (S_2 * P))$

[[10, Proposition 3.10]: $*$ distributes over $\Cup$ of outer deterministic S_i in P]

$= ((S_1 * S_1) \cup (S_1 * S_2) \cup ((S_1 * S_1) \Cup (S_1 * S_2))) \Cup ((S_2 * S_1) \cup (S_2 * S_2) \cup ((S_2 * S_1) \Cup (S_2 * S_2)))$

[simplification since S_i is deterministic]

$= (S_2 \cup S_3 \cup (S_2 \Cup S_3)) \Cup (S_3 \cup S_4 \cup (S_3 \Cup S_4))$

[Theorem 3.2: $\Cup$ distributes over $\cup$, $\Cup$ is associative, $\Cup$ of union-closed S_i is idempotent]

$= (S_2 \Cup S_3) \cup (S_2 \Cup S_4) \cup (S_2 \Cup S_3 \Cup S_4) \cup S_3 \cup (S_3 \Cup S_4)$

This allows the algebraic analysis of the composition of the players' moves. For example, it can be seen that one of the choices in the result is the inner deterministic S_3, which means that the second player can guarantee this particular outcome irrespective of the choice of the first player.

4. Inner closures

We have mentioned in Example 3.5 that the fixpoints of $(-) \Cup U$ are the up-closed multirelations [14]. These play an important role in the semantics of game logics. The inner isomorphism induces of course a dual notion of down-closure. We define these notions, add a notion of convex-closure, and discuss the subalgebras induced.

4.1. Definition of inner closures

The *(inner) up-closure*, *down-closure* and *convex-closure* of $R : X \twoheadrightarrow \mathcal{P}Y$ are defined as

$$R{\uparrow} = R \Cup U, \qquad R{\downarrow} = R \Cap U, \qquad R{\updownarrow} = R{\uparrow} \cap R{\downarrow}.$$

It is straightforward to check that $(-){\uparrow}$, $(-){\downarrow}$ and $(-){\updownarrow}$ are indeed closure operators (recall that U is an idempotent of $\Cup$ and $\Cap$). The subsets of up-, down- and convex-closed multirelations in $M(X,Y)$ can thus be defined in terms of fixpoints:

$$\begin{aligned} M_{\uparrow}(X,Y) &= \{\, R \mid R \Cup U = R \,\}, \\ M_{\downarrow}(X,Y) &= \{\, R \mid R \Cap U = R \,\}, \\ M_{\updownarrow}(X,Y) &= \{\, R \mid R = R{\updownarrow} \,\}. \end{aligned}$$

Alternatively, we can use the subset relation Ω, introduced in Section 2.1, to define $R{\uparrow} = R\Omega$ and $R{\downarrow} = R\Omega^{\smile}$. Expanding definitions,

$$\begin{aligned} R{\uparrow} &= \{\, (a, A) \mid \exists (a, B) \in R.\ B \subseteq A \,\}, \\ R{\downarrow} &= \{\, (a, A) \mid \exists (a, B) \in R.\ A \subseteq B \,\}, \\ R{\updownarrow} &= \{\, (a, A) \mid \exists (a, B), (a, C) \in R.\ B \subseteq A \subseteq C \,\}. \end{aligned}$$

As noted in the introduction, inner-closed multirelations offer greater flexibility with inner choices. Up-closed multirelations allow weakening inner choices in that one can always add options to any given set of inner choices. Likewise, with down-closed multirelations one can always strengthen inner choices by disregarding options in any given set. Convex-closed multirelations therefore enable any range of inner choices bounded by any two inner sets in the multirelation.

Up-closure and down-closure are indeed related by duality; we need this fact in Theorem 4.4 below.

Lemma 4.1. *Let $R : X \twoheadrightarrow \mathcal{P}Y$. Then $\sim(R{\uparrow}) = ({\sim}R){\downarrow}$, $\sim(R{\downarrow}) = ({\sim}R){\uparrow}$ and $\sim(R{\updownarrow}) = ({\sim}R){\updownarrow}$.*

Remark 4.2. The relationship $(a, B) \in 1_X \Cup U_{X,\mathcal{P}X}$ if and only if $a \in B$ confirms that $\in\, = 1{\uparrow}$ can be defined in the multirelational language. See Appendix A for context.

4.2. Structure of inner-closed sets

The inner-closed multirelations form quantales similar to those in Theorem 3.2, but part of the inner structure collapses: $\Cup$ becomes $\cap$ when multirelations are up-closed [15]; dually, therefore, $\Cap$ becomes $\cap$ when they are down-closed. First we note the following fact without proof.

Lemma 4.3. *Up- and down-closure of multirelations preserve arbitrary unions:*

$$\Big(\bigcup S\Big){\uparrow} = \bigcup_{R \in S} R{\uparrow} \qquad \textit{and} \qquad \Big(\bigcup S\Big){\downarrow} = \bigcup_{R \in S} R{\downarrow}.$$

These operations need not preserve intersections, but arbitrary intersections of closed elements of any closure operator are of course closed.

Next we present a refinement of Theorem 3.2. It uses the operations $Q \Cup\!\updownarrow R = (Q \Cup R)\!\updownarrow$ and $Q \Cap\!\updownarrow R = (Q \Cap R)\!\updownarrow$ for multirelations $Q, R : X \nrightarrow \mathcal{P}Y$.

Theorem 4.4.

1. $(M_{\downarrow}(X,Y), \subseteq, \Cup, 1_{\Cup})$ *is a commutative subquantale of* $M_{\Cup}(X,Y)$ *in which* $\Cap = \cap$, $1_{\Cap}\!\downarrow = U$ *and* $1_{\Cup} = 1_{\Cup}\!\downarrow$.
2. $(M_{\uparrow}(X,Y), \subseteq, \Cap, 1_{\Cap})$ *is a commutative subquantale of* $M_{\Cap}(X,Y)$ *in which* $\Cup = \cap$, $1_{\Cup}\!\uparrow = U$ *and* $1_{\Cap} = 1_{\Cap}\!\uparrow$.
3. *Maps* $(-)\!\downarrow : M(X,Y) \to M_{\downarrow}(X,Y)$ *and* $(-)\!\uparrow : M(X,Y) \to M_{\uparrow}(X,Y)$ *are quantale homomorphisms,* $\sim : M_{\downarrow}(X,Y) \to M_{\uparrow}(X,Y)$ *is a quantale isomorphism.*
4. $(M_{\updownarrow}(X,Y), \subseteq)$ *is a complete lattice in which* $\inf = \bigcap$ *and* $\sup S = (\bigcup S)\!\updownarrow$*. It forms commutative quantales* $M_{\Cup\updownarrow}(X,Y) = (M_{\updownarrow}(X,Y), \subseteq, \Cup\!\updownarrow, 1_{\Cup})$ *and* $M_{\Cap\updownarrow}(X,Y) = (M_{\updownarrow}(X,Y), \subseteq, \Cap\!\updownarrow, 1_{\Cap})$.
5. *The map* $(-)\!\updownarrow : M(X,Y) \to M_{\updownarrow}(X,Y)$ *is a quantale homomorphism from* $M_{\Cup}(X,Y)$ *to* $M_{\Cup\updownarrow}(X,Y)$ *and from* $M_{\Cap}(X,Y)$ *to* $M_{\Cap\updownarrow}(X,Y)$*, respectively. The map* $\sim : M_{\updownarrow}(X,Y) \to M_{\updownarrow}(X,Y)$ *is a quantale isomorphism between* $M_{\Cup\updownarrow}(X,Y)$ *and* $M_{\Cap\updownarrow}(X,Y)$.

Proof. For (1)–(3) note that $(-)\!\downarrow$ and $(-)\!\uparrow$ are nuclei: closure operators satisfying $R\!\downarrow \Cup S\!\downarrow \subseteq (R \Cup S)\!\downarrow$ and $R\!\uparrow \Cap S\!\uparrow \subseteq (R \Cap S)\!\uparrow$, and in fact

$$R\!\downarrow \Cup S\!\downarrow = (R \Cup S)\!\downarrow \qquad \text{and} \qquad R\!\uparrow \Cap S\!\uparrow = (R \Cap S)\!\uparrow.$$

Hence, by standard theory, $M_{\downarrow}(X,Y)$ forms a quantale with composition $\Cup$ and the map $(-)\!\downarrow : M(X,Y) \to M_{\downarrow}(X,Y)$ is a quantale homomorphism. Likewise $M_{\uparrow}(X,Y)$ is a quantale with composition $\Cap$ and $(-)\!\uparrow : M(X,Y) \to M_{\uparrow}(X,Y)$ is a quantale homomorphism [31, Theorem 3.3.1]. Moreover, $1_{\Cup}\!\downarrow = 1_{\Cup}$ and $1_{\Cap}\!\uparrow = 1_{\Cap}$ show unit preservation. The map $\sim$ is a quantale isomorphism by Theorem 3.2 and Lemma 4.1. Further,

$$(R \Cup S)\!\uparrow = R\!\uparrow \Cup S\!\uparrow = R\!\uparrow \cap S\!\uparrow \qquad \text{and} \qquad (R \Cap S)\!\downarrow = R\!\downarrow \Cap S\!\downarrow = R\!\downarrow \cap S\!\downarrow.$$

For $\Cup$, this fact is known [15]. That for $\Cap$ then follows from inner duality. Idempotency of $\Cup$ for up-closed multirelations and of $\Cap$ for down-closed multirelations and coincidence with $\cap$ are trivial consequences of these facts.

For (4), $M_{\updownarrow}(X,Y)$ is a complete lattice in which inf is intersection and sup is convex-closure of union because $(-)\!\updownarrow$ is a closure operation [8, Proposition 7.2]. The monoid structure of $\Cup\!\updownarrow$ follows from that of $\Cup$ by

$$(Q \Cup\!\updownarrow R) \Cup\!\updownarrow S = (Q \Cup R) \Cup\!\updownarrow S = Q \Cup\!\updownarrow (R \Cup S) = Q \Cup\!\updownarrow (R \Cup\!\updownarrow S).$$

This uses the laws $Q \Cup\!\updownarrow R = Q\!\updownarrow \Cup\!\updownarrow R = Q \Cup\!\updownarrow R\!\updownarrow$ to apply convex closure in operands of $\Cup\!\updownarrow$. These laws are consequences of $Q \Cup\!\updownarrow R = (Q\!\downarrow \Cup R\!\downarrow) \cap Q\!\uparrow \cap R\!\uparrow$, which follows from properties mentioned above. It remains to show that $\Cup\!\updownarrow$ distributes over sups. To this end, observe that

$$\left(\bigcup S\right)\!\updownarrow \Cup\!\updownarrow Q = \left(\bigcup S\right) \Cup\!\updownarrow Q = \left(\bigcup_{R \in S} R \Cup Q\right)\!\updownarrow \subseteq \left(\bigcup_{R \in S} R \Cup\!\updownarrow Q\right)\!\updownarrow.$$

For the converse inclusion, $R \Cup\!\updownarrow Q \subseteq \left(\bigcup_{R \in S} R \Cup Q\right)\!\updownarrow$ for each $R \in S$ since $(-)\!\updownarrow$ preserves $\subseteq$. Hence $\bigcup_{R \in S} R \Cup\!\updownarrow Q \subseteq \left(\bigcup_{R \in S} R \Cup Q\right)\!\updownarrow$, which implies $\left(\bigcup_{R \in S} R \Cup\!\updownarrow Q\right)\!\updownarrow \subseteq \left(\bigcup_{R \in S} R \Cup Q\right)\!\updownarrow$ since $(-)\!\updownarrow$ is a closure operation. The proof for $\Cap\!\updownarrow$ is similar to that for $\Cup\!\updownarrow$.

For (5), both inclusions of the equality

$$\left(\bigcup S\right)\!\updownarrow = \left(\bigcup_{R \in S} R\!\updownarrow\right)\!\updownarrow$$

follow since $(-)\!\updownarrow$ is a closure operation. Preservation of the monoid operation follows using $Q \Cup\!\updownarrow R = Q\!\updownarrow \Cup\!\updownarrow R\!\updownarrow$ and $Q \Cap\!\updownarrow R = Q\!\updownarrow \Cap\!\updownarrow R\!\updownarrow$. The automorphism claim about $\sim$ follows from Theorem 3.2 and Lemma 4.1 since $\sim$ preserves arbitrary unions and intersections. $\square$

The complete sublattices of $M(X,Y)$ need not be boolean: $M_\uparrow(X,Y)$, $M_\downarrow(X,Y)$ and $M_\updownarrow(X,Y)$ are not closed under complementation.

Example 4.5. The inner intersection of down-closed multirelations, as set-intersection, is idempotent. Yet the inner union of down-closed multirelations need not be idempotent: for the multirelation R in Example 3.4,

$$R\!\downarrow = \{(a,\emptyset),(a,\{a\}),(a,\{b\})\} \subset R\!\downarrow \cup \{(a,\{a,b\})\} = R\!\downarrow \Cup R\!\downarrow.$$

Dually, while the inner union of up-closed multirelations is idempotent, the inner intersection of up-closed multirelations need not be idempotent: assuming that R is a multirelation on $\{a,b\}$, $R\!\uparrow = R \Cup R \subset (R \Cup R) \cup \{(a,\emptyset)\} = R\!\uparrow \Cap R\!\uparrow$. This shows that set inclusion is still not the natural order on $M_\uparrow(X,Y)$ and $M_\downarrow(X,Y)$. Note, however, that $R\!\uparrow \cap S\!\uparrow \subseteq R\!\uparrow \Cap S\!\uparrow$ and $R\!\downarrow \cap S\!\downarrow \subseteq R\!\downarrow \Cup S\!\downarrow$.

4.3. Inner closures and Peleg composition

The inner operations, in particular up-closure, have so far been studied primarily in combination with Parikh's composition of multirelations in game logics [25, 27], which is relational composition with the Parikh lifting $\in\backslash-$ [12].

Recall that multirelations under Peleg composition and the outer operations do not form typed quantales – or quantaloids – because Peleg composition is not associative and does not preserve all sups in its second argument. For similar reasons, and the failure of idempotency of inner union and intersection, multirelations under Peleg composition and the inner operations do not form quantaloids. See [14, 15] for more details on these structures. Here, instead, we relate the inner operations with Peleg composition, which leads to an alternative characterisation of down-closure for multirelations.

Proposition 4.6. *Let $R : X \nrightarrow \mathcal{P}Y$. Then $R\!\downarrow = R * 1\!\downarrow$; hence the down-closed multirelations are the fixpoints of $(-) * 1\!\downarrow$:*

$$M_\downarrow(X,Y) = \{\, R : X \nrightarrow \mathcal{P}Y \mid R * 1\!\downarrow = R \,\}.$$

Proof. Simple set-theoretic reasoning shows that $(1\!\downarrow)_* = \Omega^\smile$ (we currently do not know an algebraic proof). Thus $R\!\downarrow = R\Omega^\smile = R(1\!\downarrow)_* = R * 1\!\downarrow$. □

The following fact is then immediate.

Proposition 4.7. *Peleg composition preserves down-closure of multirelations.*

Proof. Suppose R and S are composable multirelations. Then $(R*S)\!\downarrow = (R*S)*1\!\downarrow = R*(S*1\!\downarrow) = R*S\!\downarrow$ by Proposition 4.6. The second equality uses that Peleg composition of three multirelations is associative if the third multirelation, in this instance $1\!\downarrow$, is union-closed [12]. Finally, $(R\!\downarrow * S\!\downarrow)\!\downarrow = R\!\downarrow * S\!\downarrow$ follows immediately. □

Example 4.8. One might wonder whether down-closed multirelations form categories with respect to Peleg composition. The answer is negative: the multirelations on $X = \{a,b\}$ given by

$$R = \{(a,\emptyset),(a,\{a\}),(a,\{b\}),(a,X)\},$$
$$S = \{(a,\emptyset),(a,\{b\}),(b,\emptyset),(b,\{b\})\},$$
$$T = \{(b,\emptyset),(b,\{a\}),(b,\{b\})\}$$

are down-closed, but $(R*S)*T = \{(a,\emptyset),(a,\{a\}),(a,\{b\})\} \subset R = R*(S*T)$.

This raises the question whether similar properties hold with respect to up-closure. The following example shows that further restrictions need to be imposed.

Example 4.9. Peleg compositions of general up-closed multirelations need not be up-closed: $1_\Cup\!\uparrow * 1_\Cap\!\uparrow = U * 1_\Cap = 1_\Cup \cup 1_\Cap \subset U = 1_\Cup\!\uparrow = (1_\Cup * 1_\Cap)\!\uparrow$. Note that both $1_\Cup$ and $1_\Cap$ are deterministic.

The following lemma, needed in Proposition 5.3(6), shows sufficient restrictions. Recall from Remark 4.2 that $\in = 1\!\uparrow$.

Lemma 4.10. *Let $R : X \nrightarrow \mathcal{P}Y$ be inner deterministic and $S : Y \nrightarrow \mathcal{P}Z$. Then*

1. $R\!\uparrow = R * 1\!\uparrow$,
2. $1\!\uparrow * S\!\uparrow = S\!\uparrow$,
3. $(R * S)\!\uparrow = R * S\!\uparrow = R\!\uparrow * S\!\uparrow$.

Proof. For (1), $R * 1\!\uparrow = R1^\smile 1\!\uparrow = R1^\smile 1\Omega = (R * 1)\Omega = R\Omega = R\!\uparrow$, by Lemma 3.10(1) and the alternative definition $R\!\uparrow = R\Omega$ of up-closure.

For (2), clearly $S\!\uparrow = 1 * S\!\uparrow \subseteq 1\!\uparrow * S\!\uparrow$. We obtain the converse inclusion using

$$1\!\uparrow * S\!\uparrow = \in dom(S)_* \bigcup_{Q \subseteq_d S} Q_{\mathcal{P}} = \bigcup_{Q \subseteq_d S} \in dom(Q)_* Q_{\mathcal{P}} \subseteq S\!\uparrow$$

if we can show $\in dom(Q)_* Q_{\mathcal{P}} \subseteq Q\!\uparrow$ for univalent Q. Since

$$\in dom(Q)_* = \in((\in\backslash dom(Q)\in) \cap Id) \subseteq \in(\in\backslash dom(Q)\in) \subseteq dom(Q)\in,$$

it remains to show $dom(Q)\in Q_{\mathcal{P}} \subseteq Q\!\uparrow$. Since $Q_{\mathcal{P}}$ is a function and Q is univalent, this is equivalent to

$$\begin{aligned}
dom(Q)\in &\subseteq Q\Omega(Q_{\mathcal{P}})^\smile && [Sf \subseteq T \iff S \subseteq Tf^\smile \text{ for function } f]\\
&= Q(\in\backslash\in(Q_{\mathcal{P}})^\smile) && [(S\backslash T)f^\smile = S\backslash(Tf^\smile) \text{ for function } f]\\
&= Q(\in\backslash\in(\in \div \in Q^\smile\in)) && [(S \div T)^\smile = T \div S]\\
&= Q(\in\backslash\in Q^\smile\in) && [\in(\in \div S) = S]\\
&= QU \cap (\in Q^\smile\backslash\in Q^\smile\in), && [e(S\backslash T) = eU \cap (Se^\smile\backslash T) \text{ for univalent } e]
\end{aligned}$$

which follows from

$$dom(Q)\in \subseteq dom(Q)U = QU \qquad \text{and} \qquad \in Q^\smile dom(Q)\in \subseteq \in Q^\smile\in.$$

For (3), Lemma 3.10, (1) and (2) imply that

$$(R * S)\!\uparrow = R1^\smile S\Omega = R * S\!\uparrow = R * (1\!\uparrow * S\!\uparrow) = (R * 1\!\uparrow) * S\!\uparrow = R\!\uparrow * S\!\uparrow. \qquad \square$$

Lemma 4.10(1) is an analogue of the identity in Proposition 4.6, but does not describe general multirelations. Lemma 4.10(2) is needed in the proof of (3). Lemma 4.10(3) is an analogue of Proposition 4.7.

The up-closure of the Peleg composition of up-closed multirelations equals their Parikh composition [15] (and the co-composition of up-closed multirelations is up-closed, see Section 6). As up-closed multirelations are union-closed, their Peleg composition is associative [12]. Yet the units of Peleg composition are not up-closed, so that up-closed multirelations do not form categories under Peleg composition.

Example 4.11. The property $(R * S)\!\uparrow = R\!\uparrow * S\!\uparrow$ from Lemma 4.10(3) does not translate to down-closure:

$$(1 * \emptyset)\!\downarrow = \emptyset\!\downarrow = \emptyset \subset 1_{\Cup} = 1_{\Cup} * \emptyset = (1 \cup 1_{\Cup}) * \emptyset = 1\!\downarrow * \emptyset\!\downarrow.$$

Note that both 1 and $\emptyset$ are inner deterministic.

5. Inner preorders

Example 3.4 shows that $\subseteq$ is not the natural order for $\Cup$ and $\Cap$. In fact, $R \Cup S = R$ if and only if for each $R_{a,B}$ there is a $S_{a,C}$ with $C \subseteq B$ and $R_{a,B\cup C}$ holds for each $R_{a,B}$ and $S_{a,C}$. Theorem 4.4 shows that restrictions to up- or down-closed relations collapse part of the inner structure. It is standard to define preorders, equivalences and partial orders based on the inclusion of closed sets. Here, these preorders compare the inner nondeterminism of multirelations in different ways, while set inclusion obviously compares their outer nondeterminism. Apart from the obvious interest in such comparisons, this raises the question whether these orders are natural for inner union and inner intersection. The general answer is negative.

5.1. Definition of inner preorders

For $R, S : X \nrightarrow \mathcal{P}Y$, we define the *Smyth preorder* $\sqsubseteq_\uparrow$ [29], its dual *Hoare preorder* $\sqsubseteq_\downarrow$ and the *Egli-Milner preorder* $\sqsubseteq_\updownarrow$ as

$$R \sqsubseteq_\uparrow S \Leftrightarrow S \subseteq R\!\uparrow, \qquad R \sqsubseteq_\downarrow S \Leftrightarrow R \subseteq S\!\downarrow, \qquad R \sqsubseteq_\updownarrow S \Leftrightarrow R \sqsubseteq_\downarrow S \wedge R \sqsubseteq_\uparrow S.$$

Expanding definitions,

$$R \sqsubseteq_\uparrow S \Leftrightarrow (\forall a, C.\ S_{a,C} \Rightarrow \exists B.\ R_{a,B} \wedge B \subseteq C),$$
$$R \sqsubseteq_\downarrow S \Leftrightarrow (\forall a, B.\ R_{a,B} \Rightarrow \exists C.\ S_{a,C} \wedge B \subseteq C).$$

Intuitively, therefore, $R \sqsubseteq_\uparrow S$ if for every outer choice of a set from a given element with S there is a less nondeterministic outer choice from that element with R. Moreover, $R \sqsubseteq_\downarrow S$ if for every outer choice of a set from a given element with R there is a more nondeterministic outer choice from that element with S.

The Smyth, Hoare and Egli-Milner preorders originate in domain theory where they are used to define the semantics of recursive programs. They describe different ways of modelling nondeterminism using power domains; see [35] for details. Their consideration in the context of multirelations is therefore natural.

The following fact is standard.

Lemma 5.1. *Let $R, S : X \nrightarrow \mathcal{P}Y$. Then*

$$R \sqsubseteq_\downarrow S \Leftrightarrow R\!\downarrow \subseteq S\!\downarrow \Leftrightarrow R\!\downarrow = R\!\downarrow \cap S\!\downarrow = (R \Cap S)\!\downarrow,$$
$$R \sqsubseteq_\uparrow S \Leftrightarrow S\!\uparrow \subseteq R\!\uparrow \Leftrightarrow S\!\uparrow = R\!\uparrow \cap S\!\uparrow = (R \Cup S)\!\uparrow.$$

However, $R\!\updownarrow \subseteq S\!\updownarrow \Leftrightarrow R \subseteq S\!\updownarrow \Leftrightarrow R \sqsubseteq_\downarrow S \sqsubseteq_\uparrow R$.

Example 5.2. While $R = R \Cap S$ thus implies $R \sqsubseteq_\downarrow S$ and $S = R \Cup S$ implies $R \sqsubseteq_\uparrow S$, the converse implications, which would be typical for natural orders, do not hold: for $X = \{a\}$, $R = \{(a, \emptyset)\}$, $S = \{(a, \{a\})\}$ and $T = R \cup S$ satisfy $S \sqsubseteq_\downarrow T$ and $T \sqsubseteq_\uparrow R$, but $S \Cap T = T \neq S$ and $T \Cup R = T \neq R$.

We associate equivalences $=_\downarrow$, $=_\uparrow$ and $=_\updownarrow$ with $\sqsubseteq_\downarrow$, $\sqsubseteq_\uparrow$ and $\sqsubseteq_\updownarrow$ in the standard way by intersecting the preorders with their converses. Thus

$$R =_\downarrow S \Leftrightarrow R\!\downarrow = S\!\downarrow, \qquad R =_\uparrow S \Leftrightarrow R\!\uparrow = S\!\uparrow, \qquad R =_\updownarrow S \Leftrightarrow R \sqsubseteq_\updownarrow S \wedge S \sqsubseteq_\updownarrow R.$$

It follows that $R =_\updownarrow S \Leftrightarrow R\!\updownarrow = S\!\updownarrow$ and therefore $R =_\updownarrow R\!\updownarrow$.

5.2. Algebras of preordered multirelations

The following results describe the structure of the preorders and the resulting quotient quantales.

Proposition 5.3.

1. *$(M(X,Y), \sqsubseteq_\downarrow, \Cup, 1_\Cup)$ and $(M(X,Y), \sqsubseteq_\downarrow, \Cap, 1_\Cap)$ are preordered commutative monoids with least element $\emptyset$ and greatest element U.*
2. *$(M(X,Y), \sqsubseteq_\uparrow, \Cup, 1_\Cup)$ and $(M(X,Y), \sqsubseteq_\uparrow, \Cap, 1_\Cap)$ are preordered commutative monoids with least element U and greatest element $\emptyset$.*
3. *$(M(X,Y), \sqsubseteq_\updownarrow, \Cup, 1_\Cup)$ and $(M(X,Y), \sqsubseteq_\updownarrow, \Cap, 1_\Cap)$ are preordered commutative monoids.*
4. *$\sim$ is an order-reversing preordered monoid isomorphism:*

$$R \sqsubseteq_\downarrow S \Leftrightarrow {\sim}S \sqsubseteq_\uparrow {\sim}R, \qquad R \sqsubseteq_\uparrow S \Leftrightarrow {\sim}S \sqsubseteq_\downarrow {\sim}R, \qquad R \sqsubseteq_\updownarrow S \Leftrightarrow {\sim}S \sqsubseteq_\updownarrow {\sim}R.$$

5. *The operations $\cup$, $\uparrow$, $\downarrow$ and $\updownarrow$ preserve $\sqsubseteq_\downarrow$, $\sqsubseteq_\uparrow$ and $\sqsubseteq_\updownarrow$.*
6. *Peleg composition preserves $\sqsubseteq_\downarrow$, $\sqsubseteq_\uparrow$ and $\sqsubseteq_\updownarrow$ in its second argument.*

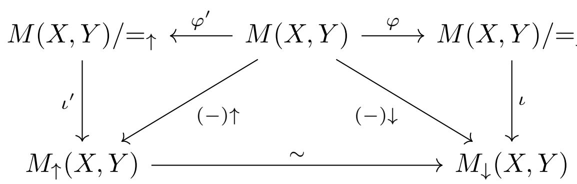
Item (6) follows from Proposition 4.7 and Lemma 4.10(3) because Peleg composition and $\Cup$ preserve $\subseteq$ in their second arguments. In domain theory, order-preservation of program and specification constructs is important for compositionality and the fixpoint semantics of recursion.

Remark 5.4. Similarly, the three equivalences $=_\downarrow$, $=_\uparrow$ and $=_\updownarrow$ are congruences with respect to $\cup$, $\Cup$, $\Cap$, $\uparrow$, $\downarrow$, $\updownarrow$ and $\sim$. Peleg composition preserves them in its second argument. The inner isomorphism $\sim$ satisfies $R =_\downarrow S \Leftrightarrow {\sim}R =_\uparrow {\sim}S$, $R =_\uparrow S \Leftrightarrow {\sim}R =_\downarrow {\sim}S$ and $R =_\updownarrow S \Leftrightarrow {\sim}R =_\updownarrow {\sim}S$. Unlike $\sqsubseteq_\downarrow$ and $\sqsubseteq_\uparrow$, $\sqsubseteq_\updownarrow$ has no least or greatest element.

Theorem 5.5.

1. $(M(X,Y)/{=_\downarrow}, \leq_{\mathrm{H}}, \Cup_{\mathrm{H}}, 1_{\Cup_{\mathrm{H}}})$, *with* $[R] \leq_{\mathrm{H}} [S] \Leftrightarrow R{\downarrow} \subseteq S{\downarrow}$, $[R] \Cup_{\mathrm{H}} [S] = [R \Cup S]$, $1_{\Cup_{\mathrm{H}}} = \{1_\Cup\}$, $[R] \Cap_{\mathrm{H}} [S] = [R{\downarrow} \cap S{\downarrow}]$ *and* $1_{\Cap_{\mathrm{H}}} = \{U\}$, *is isomorphic to* $M_\downarrow(X,Y)$.
2. $(M(X,Y)/{=_\uparrow}, \leq_{\mathrm{S}}, \Cap_{\mathrm{S}}, 1_{\Cap_{\mathrm{S}}})$, *with* $[R] \leq_{\mathrm{S}} [S] \Leftrightarrow R{\uparrow} \supseteq S{\uparrow}$, $[R] \Cup_{\mathrm{S}} [S] = [R{\uparrow} \cap S{\uparrow}]$, $1_{\Cup_{\mathrm{S}}} = \{U\}$, $[R] \Cap_{\mathrm{S}} [S] = [R \Cap S]$ *and* $1_{\Cap_{\mathrm{S}}} = \{1_\Cap\}$, *is isomorphic to* $M_\uparrow(X,Y)$.
3. $(M(X,Y)/{=_\updownarrow}, \leq_{\mathrm{EM}}, \Cup_{\mathrm{EM}}, 1_{\Cup_{\mathrm{EM}}})$, *with* $[R] \leq_{\mathrm{EM}} [S] \Leftrightarrow R{\updownarrow} \subseteq S{\updownarrow}$, $[R] \Cup_{\mathrm{EM}} [S] = [R \Cup S]$, $1_{\Cup_{\mathrm{EM}}} = \{1_\Cup\}$, $[R] \Cap_{\mathrm{EM}} [S] = [R \Cap S]$ *and* $1_{\Cap_{\mathrm{EM}}} = \{1_\Cap\}$, *is isomorphic to* $M_{\Cup\updownarrow}(X,Y)$, *and* $(M(X,Y)/{=_\updownarrow}, \leq_{\mathrm{EM}}, \Cap_{\mathrm{EM}}, 1_{\Cap_{\mathrm{EM}}})$ *is isomorphic to* $M_{\Cap\updownarrow}(X,Y)$.
4. ${\sim}[R]_{\mathrm{H}} = [{\sim}R]_{\mathrm{S}}$, ${\sim}[R]_{\mathrm{S}} = [{\sim}R]_{\mathrm{H}}$ *and* ${\sim}[R]_{\mathrm{EM}} = [{\sim}R]_{\mathrm{EM}}$.

Proof. The following diagram illustrates the construction in (1) and (2).



By Theorem 4.4, $(-){\downarrow} : M(X,Y) \to M_\downarrow(X,Y)$ is a quantale homomorphism. It follows from standard results of universal algebra [6, Theorem 6.8] that its kernel, $=_\downarrow$, is a congruence that preserves the quantale operations. The associated quotient algebra $M(X,Y)/{=_\downarrow}$ is an algebra with the same signature and quantale operations defined as in (1). The natural map $\varphi : M(X,Y) \to M(X,Y)/{=_\downarrow}$, which associates each element with its equivalence class, is thus a bijective quantale homomorphism [6, Theorem 6.10]. By [6, Theorem 6.12], there is then an isomorphism $\iota : M(X,Y)/{=_\downarrow} \to M_\downarrow(X,Y)$, here given by $\iota : [R] \mapsto R{\downarrow}$, such that the above diagram commutes.

The order isomorphism between $\leq_{\mathrm{H}}$ and $\subseteq$ is established by the fact that $[R] \leq_{\mathrm{H}} [S] \Leftrightarrow R \sqsubseteq_\downarrow S$ (by definition of $\sqsubseteq_\downarrow$) and by Lemma 5.1. It remains to consider inner intersection $\Cap_{\mathrm{H}}$ and its unit. The isomorphism sends the inner intersection $[R] \Cap_{\mathrm{H}} [S] = [R \Cap S]$ to $(R \Cap S){\downarrow} = R{\downarrow} \Cap S{\downarrow} = R{\downarrow} \cap S{\downarrow}$. Finally, the associated unit is mapped to U.

The proofs for (2) and (3) are similar. Finally, (4) is obvious. □

Remark 5.6. By construction, $R \Cup R =_\downarrow R \cap R = R$ and $R \Cap R =_\uparrow R \cap R = R$ due to the collapse of structure. Nevertheless, $R \Cap R =_\downarrow R \cap R = R$ and $R \Cup R =_\uparrow R \cap R = R$ need not hold. This is a consequence of Examples 3.4 and 4.5, recalling that $(R \Cup S){\downarrow} = R{\downarrow} \Cup S{\downarrow}$ and dually $(R \Cap S){\uparrow} = R{\uparrow} \Cap S{\uparrow}$.

The question thus remains whether $\sqsubseteq_\downarrow$ and $\sqsubseteq_\uparrow$ are natural orders on certain subalgebras of $M(X,Y)$. We provide an answer in the next section.

We conclude this section with a collection of properties that are needed in the following sections, proved using Isabelle.

Lemma 5.7. *Let* $R, S, T : X \nrightarrow \mathcal{P}Y$. *Then*

1. $R \subseteq S \Rightarrow R \cap T \sqsubseteq_\downarrow S \cap T \sqsubseteq_\uparrow R \cap T$,
2. $R \Cap S \sqsubseteq_\downarrow R \sqsubseteq_\downarrow R \Cup R$, $R \Cap R \sqsubseteq_\uparrow R \sqsubseteq_\uparrow R \Cup S$ *and* $R \Cap R \sqsubseteq_\updownarrow R \sqsubseteq_\updownarrow R \Cup R$,
3. $R \Cap S \sqsubseteq_\downarrow R \Cup S$, $R \Cap S \sqsubseteq_\uparrow R \Cup S$ *and* $R \Cap S \sqsubseteq_\updownarrow R \Cup S$,

4. *$R \Cap S$ is the inf and $R \cup S$ the sup of R and S, up-to $=_\downarrow$, with respect to $\sqsubseteq_\downarrow$,*
5. *$R \Cup S$ is the sup and $R \cup S$ the inf of R and S, up-to $=_\uparrow$, with respect to $\sqsubseteq_\uparrow$.*

Items (4) and (5) do not imply that $\sqsubseteq_\downarrow$ and $\sqsubseteq_\uparrow$ are orders as they only provide infs and sups up-to the equivalences $=_\downarrow$ and $=_\uparrow$, respectively, in line with Theorem 5.5.

5.3. Inner preorders on special multirelations

In this section, we consider $\sqsubseteq_\downarrow$, $\sqsubseteq_\uparrow$ and $\sqsubseteq_\updownarrow$ on subclasses of multirelations. First we consider cases for which these preorders become partial orders.

Proposition 5.8.

1. *On inner deterministic multirelations, the preorders $\sqsubseteq_\downarrow$ and $\sqsubseteq_\uparrow$ coincide with $\subseteq$ and $\supseteq$, respectively, whence $\sqsubseteq_\updownarrow$ is the discrete order.*
2. *The preorder $\sqsubseteq_\updownarrow$ is a partial order on inner univalent multirelations.*
3. *The preorders $\sqsubseteq_\downarrow$, $\sqsubseteq_\uparrow$, $\sqsubseteq_\updownarrow$ are partial orders on univalent multirelations.*
4. *The three partial orders coincide on deterministic multirelations.*

Proof. Note that

$$1(Id \cup -\Omega^\smile)1^\smile = 11^\smile \cup 1(-\Omega^\smile)1^\smile = Id \cup -(1\Omega^\smile 1^\smile) = Id \cup -(1\in^\smile) = Id \cup -Id = U.$$

Hence $\mathsf{A}_\Cup{}^\smile \mathsf{A}_\Cup = 1^\smile U 1 = 1^\smile 1(Id \cup -\Omega^\smile)1^\smile 1 \subseteq Id \cup -\Omega^\smile$, and further $\Omega^\smile \cap \mathsf{A}_\Cup{}^\smile \mathsf{A}_\Cup \subseteq Id$.

For (1), assume that $R \sqsubseteq_\downarrow S$ for inner deterministic R and S. Then

$$R = R \cap \mathsf{A}_\Cup \subseteq S\!\downarrow \cap \mathsf{A}_\Cup = (S \cap \mathsf{A}_\Cup)\Omega^\smile \cap \mathsf{A}_\Cup = S(\Omega^\smile \cap \mathsf{A}_\Cup{}^\smile \cap \mathsf{A}_\Cup) = S(\Omega^\smile \cap \mathsf{A}_\Cup{}^\smile \mathsf{A}_\Cup) \subseteq S.$$

The converse implication follows by $R \subseteq S \subseteq S\!\downarrow$. Moreover, from $R \sqsubseteq_\uparrow S$ we obtain

$$S = S \cap \mathsf{A}_\Cup \subseteq R\!\uparrow \cap \mathsf{A}_\Cup = (R \cap \mathsf{A}_\Cup)\Omega \cap \mathsf{A}_\Cup = R(\Omega \cap \mathsf{A}_\Cup{}^\smile \cap \mathsf{A}_\Cup) = R(\Omega \cap \mathsf{A}_\Cup{}^\smile \mathsf{A}_\Cup) \subseteq R.$$

The converse implication follows by $S \subseteq R \subseteq R\!\uparrow$.

For (2), we prove antisymmetry of $\sqsubseteq_\updownarrow$ in the inner univalent case. Suppose $R \sqsubseteq_\updownarrow S$ and $S \sqsubseteq_\updownarrow R$ for inner univalent R and S. We show $R \subseteq S$. The assumption implies that

$$R = R \cap S\!\downarrow \cap S\!\uparrow = R \cap S\Omega^\smile \cap S\Omega \subseteq S(\Omega^\smile \cap S^\smile R) \cap S\Omega.$$

Since R and S are inner univalent, we have

$$S^\smile R \subseteq (1_\Cup \cup \mathsf{A}_\Cup)^\smile (1_\Cup \cup \mathsf{A}_\Cup) \subseteq U1_\Cup \cup 1_\Cup{}^\smile \mathsf{A}_\Cup \cup \mathsf{A}_\Cup{}^\smile \mathsf{A}_\Cup.$$

Hence, by distributivity, it suffices to consider the following three cases. First,

$$S(\Omega^\smile \cap U1_\Cup) \cap S\Omega \subseteq U1_\Cup \cap S\Omega = S(\Omega \cap U1_\Cup) \subseteq S\Omega 1_\Cup{}^\smile 1_\Cup = S(1_\Cup\!\downarrow)^\smile 1_\Cup = S1_\Cup{}^\smile 1_\Cup \subseteq S$$

using $1_\Cup{}^\smile 1_\Cup \subseteq Id$. Second, $S(\Omega^\smile \cap 1_\Cup{}^\smile \mathsf{A}_\Cup) \cap S\Omega = \emptyset \subseteq S$ using

$$\Omega^\smile \cap 1_\Cup{}^\smile \mathsf{A}_\Cup \subseteq 1_\Cup{}^\smile(\mathsf{A}_\Cup \cap 1_\Cup \Omega^\smile) \subseteq U(\mathsf{A}_\Cup \cap 1_\Cup\!\downarrow) = U(\mathsf{A}_\Cup \cap 1_\Cup) = U\emptyset = \emptyset.$$

Third, $S(\Omega^\smile \cap \mathsf{A}_\Cup{}^\smile \mathsf{A}_\Cup) \cap S\Omega \subseteq SId = S$. The proof of $S \subseteq R$ follows along similar lines.

For (3), we first prove antisymmetry of $\sqsubseteq_\downarrow$. Suppose $R \sqsubseteq_\downarrow S$ and $S \sqsubseteq_\downarrow R$, that is, $R =_\downarrow S$, for univalent R and S. Then $S^\smile R \subseteq S^\smile S\!\downarrow = S^\smile S\Omega^\smile \subseteq \Omega^\smile$ and likewise $R^\smile S \subseteq \Omega^\smile$ by univalence of R and S. Thus $S^\smile R \subseteq \Omega^\smile \cap \Omega = Id$. Also, $R = R \cap S\!\downarrow = R \cap S\Omega^\smile \subseteq SS^\smile R \subseteq S$ and $S \subseteq R$ follows by opposition. This proves $R = S$.

Antisymmetry of $\sqsubseteq_\uparrow$ is proved along similar lines. Antisymmetry of $\sqsubseteq_\updownarrow$ is then immediate.

For (4), suppose R and S are deterministic. Then $\sqsubseteq_\downarrow$ and $\sqsubseteq_\uparrow$ coincide because $R \subseteq S\!\downarrow \Leftrightarrow S^\smile \subseteq \Omega^\smile R^\smile \Leftrightarrow S \subseteq R\!\uparrow$ and the claim for $\sqsubseteq_\updownarrow$ follows. ☐

It is immediate from the proof of Proposition 5.8 that, for R, S univalent or inner deterministic,

$$R =_{\downarrow} S \Leftrightarrow R =_{\uparrow} S \Leftrightarrow R =_{\updownarrow} S \Leftrightarrow R = S.$$

Next we point out a case when $\sqsubseteq_{\downarrow}$ and $\sqsubseteq_{\uparrow}$ become natural orders.

Lemma 5.9. *Let R and S be univalent. Then*

$$R \sqsubseteq_{\uparrow} S \Leftrightarrow R \Cup S = S \qquad \text{and} \qquad R \sqsubseteq_{\downarrow} S \Leftrightarrow R \Cap S = R.$$

Proof. Suppose $R \sqsubseteq_{\downarrow} S$. Then $R \subseteq S{\downarrow}$ and hence $R \subseteq R{\downarrow} \cap S{\downarrow} = (R \Cap S){\downarrow}$ by Theorem 4.4. Thus $R \sqsubseteq_{\downarrow} R \Cap S$. By Lemma 5.7, $R \Cap S \sqsubseteq_{\downarrow} R$. Since $R \Cap S$ is univalent by Lemma 3.12, we obtain $R = R \Cap S$ by Proposition 5.8. The converse implication is immediate by Lemma 5.7.

The proof for $\sqsubseteq_{\uparrow}$ is similar. □

Proposition 5.10. *The deterministic multirelations form a lattice with respect to $\sqsubseteq_{\downarrow}$ (which is equal to $\sqsubseteq_{\uparrow}$ and $\sqsubseteq_{\updownarrow}$) with sup $\Cup$ and inf $\Cap$.*

Proof. Deterministic multirelations are closed with respect to $\Cup$ and $\Cap$ by Lemma 3.12. Since $\Cup$ and $\Cap$ are associative and commutative, it remains to verify the absorption laws. First, $R \Cup (R \Cap S) = R$ is equivalent to $R \Cap S \sqsubseteq_{\uparrow} R$ by Lemma 5.9, which is $R \Cap S \sqsubseteq_{\downarrow} R$ by Proposition 5.8, which holds by Lemma 5.7. Second, $R = R \Cap (R \Cup S)$ is equivalent to $R \sqsubseteq_{\downarrow} R \Cup S$ by Lemma 5.9, which is $R \sqsubseteq_{\uparrow} R \Cup S$ by Proposition 5.8, which holds by Lemma 5.7. □

Deterministic multirelations are isomorphic to relations, and the inner preorders allow comparing their nondeterminism.

Example 5.11. Let $X = \{a, b, c\}$ and consider $R, S : X \nrightarrow \mathcal{P}X$ with

$$R = \{(a, \{a\}), (a, \{a, b, c\})\} \qquad \text{and} \qquad S = R \cup \{(a, \{a, b\})\}.$$

Then $R =_{\downarrow} S$ and $R =_{\uparrow} S$ but $R \neq S$. Hence $\sqsubseteq_{\downarrow}$ or $\sqsubseteq_{\uparrow}$ are not partial orders on inner total multirelations. With the same example, $UR =_{\downarrow} US$ and $UR =_{\uparrow} US$ but $UR \neq US$ shows that requiring totality does not suffice either.

Moreover, on a one-element set all multirelations are inner univalent, $1 =_{\downarrow} U$ and $-1 =_{\uparrow} U$ but $1 \neq U \neq -1$. Hence inner univalence is also not enough to force a partial order.

This example also shows that $\sqsubseteq_{\updownarrow}$ is not a partial order on total or inner total multirelations.

Example 5.12. Since $\emptyset \sqsubseteq_{\downarrow} 1$ and $1 \sqsubseteq_{\uparrow} \emptyset$ but neither $\emptyset \sqsubseteq_{\uparrow} 1$ nor $1 \sqsubseteq_{\downarrow} \emptyset$ hold, the preorders $\sqsubseteq_{\downarrow}$ and $\sqsubseteq_{\uparrow}$ are incomparable for univalent, inner univalent, inner total or inner deterministic multirelations. Since $1 \sqsubseteq_{\downarrow} 1_{\Cup} \cup 1$ and $1_{\Cup} \cup 1 \sqsubseteq_{\uparrow} 1_{\Cup}$ but neither $1 \sqsubseteq_{\uparrow} 1_{\Cup} \cup 1$ nor $1_{\Cup} \cup 1 \sqsubseteq_{\downarrow} 1_{\Cup}$, the preorders $\sqsubseteq_{\downarrow}$ and $\sqsubseteq_{\uparrow}$ are incomparable for total multirelations.

Example 5.13. In the deterministic case, $\sqsubseteq_{\downarrow}$ need not coincide with $\subseteq$. For instance, $\{(a, \emptyset)\} \sqsubseteq_{\downarrow} \{(a, \{a\})\}$, but the two relations are disjoint.

5.4. Decomposition of multirelations

As an application of inner preorders, we present a decomposition theorem for multirelations. Every multirelation R can be split into its terminal and non-terminal parts: $R = \tau(R) \cup \nu(R)$ using $\tau(R) = R \cap 1_{\Cup}$ and $\nu(R) = R - 1_{\Cup}$ [15]. We write $S \sqsubseteq_{\downarrow d} R$ if S is univalent and inner deterministic, $dom(S) = dom(\nu(R))$ and $S \sqsubseteq_{\downarrow} R$. The following lemma relates this to $\subseteq_d$ using the inner determinisation map $\delta_i(R) = R{\downarrow} \cap \mathsf{A}_{\Cup}$ [10].

Lemma 5.14. *Let $R, S : X \nrightarrow \mathcal{P}Y$. Then $S \sqsubseteq_{\downarrow d} R$ if and only if $S \subseteq_d \delta_i(R)$.*

Proof. By Proposition 3.9, S is inner deterministic if and only if $S \subseteq \mathsf{A}_{\Cup}$. Moreover $S \sqsubseteq_{\downarrow} R$ if and only if $S \subseteq R{\downarrow}$. Both conditions together are equivalent to $S \subseteq \delta_i(R)$. It remains to show that $dom(\nu(R)) = dom(\delta_i(R))$, which follows from

$$\nu(R)U = R(-1_{\Cup})^{\smile} = R(\mathsf{A}_{\Cup}{\uparrow})^{\smile} = R\Omega^{\smile}\mathsf{A}_{\Cup}{}^{\smile} = R{\downarrow}\mathsf{A}_{\Cup}{}^{\smile} = \delta_i(R)U. \qquad □$$

Lemma 5.15. *Let $R : X \nrightarrow \mathcal{P}Y$ be univalent. Then $\nu(R) = \Cup_{S \sqsubseteq_{\downarrow d} R} S$ and each $S \sqsubseteq_{\downarrow d} R$ is isomorphic to a partial function from X to Y.*

Proof. Using the outer determinisation map $\delta_o(R) = \{(a, B) \mid B = \bigcup R(a)\}$ from [10],

$$\begin{aligned}
\Cup_{S \sqsubseteq_{\downarrow d} R} S &= (\bigcap_{S \sqsubseteq_{\downarrow d} R} dom(S))\delta_o(\bigcup_{S \sqsubseteq_{\downarrow d} R} S) \\
&= (\bigcap_{S \sqsubseteq_{\downarrow d} R} dom(\nu(R)))\delta_o(\bigcup_{S \subseteq_d \delta_i(R)} S) \\
&= dom(\nu(R))\delta_o(\delta_i(R)) \\
&= dom(\nu(R))\delta_o(R) \\
&= dom(\nu(R))\delta_o(\nu(R)) \\
&= \nu(R).
\end{aligned}$$

For the first step, the inner union of a family R_i of deterministic multirelations is $\Cup_{i \in I} R_i = \delta_o(\bigcup_{i \in I} R_i)$. Intuitively, if each R_i is deterministic, only one choice is possible inside $\Cup$, so all pairs can be collected using $\bigcup$ and then merged by δ_o. This generalises to $\Cup_{i \in I} R_i = (\bigcap_{i \in I} dom(R_i))\delta_o(\bigcup_{i \in I} R_i)$ for a family R_i of univalent multirelations. The intersection of domains takes care of non-total R_i, for which no choice is possible inside $\Cup$.

The second step uses the definition of $\sqsubseteq_{\downarrow d}$ and Lemma 5.14. The third step uses Lemma 2.1. The fourth and fifth steps use properties of δ_o and δ_i [10].

For the last step, we note that $R = \delta_o(R)$ for deterministic R [10]. This generalises to $R = dom(R)\delta_o(R)$ for univalent multirelations.

Finally, each $S \sqsubseteq_{\downarrow d} R$ is isomorphic to a partial function from X to Y because S is inner deterministic and the singleton sets in each pair in S correspond to elements of Y. □

This and Lemma 2.1 yield the following decomposition theorem for multirelations.

Theorem 5.16. *Let $R : X \nrightarrow \mathcal{P}Y$. Then $R = (\bigcup_{S \subseteq_d R} \Cup_{T \sqsubseteq_{\downarrow d} S} T) \cup \tau(R)$.*

Thus every multirelation R can be decomposed into an outer union of univalent multirelations, each of which, except $\tau(R)$, in turn decomposes into an inner union of inner deterministic multirelations. Such a result can be used to break down the study of multirelations into that of elementary multirelations. For example, it gives a normal form for multirelations: two multirelations are equal if and only if they have the same decomposition. Conceptually the result gives an interesting insight into the interplay of inner and outer determinism in multirelations.

Remark 5.17. Alternatively, we could define $S \sqsubseteq_{\downarrow d} R$ if S is deterministic and inner univalent and $S \sqsubseteq_{\downarrow} R$. Unlike with $\sqsubseteq_{\downarrow d}$, pairs of the form $(a, \emptyset)$ are now included. Both definitions yield a decomposition theorem, but including such pairs in decompositions is unnecessary.

6. Peleg co-composition and intersection-closure

This section discusses the notions of Peleg co-composition and intersection-closure, which are dual to Peleg composition and union-closure. The following results are summarised more compactly as a sequence of observations, but could be elaborated into lemmas and propositions similarly to the other parts of this paper. Like in the other parts, many results discussed here have been formally verified using Isabelle/HOL.

Recall the interaction of inner union and Peleg composition:

$$(R \Cup S) * T \subseteq (R * T) \Cup (S * T), \qquad R * (S \Cup T) \subseteq (R * S) \Cup (R * T),$$
$$T \Cup T \subseteq T \Rightarrow (R \Cup S) * T = (R * T) \Cup (S * T).$$

To obtain similar properties of $\Cap$ by inner duality we need to connect $\sim$ and $*$. The relationship

$$\sim(R * S) = \{ (a, C) \mid \exists B.\ R_{a,B} \wedge \exists f.\ f|_B \cap S = \emptyset \wedge C = \bigcap f(B) \}.$$

motivates defining the *Peleg co-composition*

$$R \odot S = \sim(R * \sim S) = \{ (a, C) \mid \exists B.\ R_{a,B} \wedge \exists f.\ f|_B \subseteq S \wedge C = \bigcap f(B) \}.$$

It follows immediately that $R * S = \sim(R \odot \sim S)$, $\emptyset \odot R = \emptyset = R \odot \emptyset$, $1 \odot R = R$, $\sim R = R \odot \sim 1$ and $\sim 1 \odot \sim 1 = 1$. But $\odot$ does not have a right unit because $1_{\Cup} \odot R = 1_{\Cap}$.

We also obtain $R \odot 1_{\Cup} = \sim(R * 1_{\Cap})$ and $R \odot 1_{\Cap} = \sim(R * 1_{\Cup})$ and it follows that $R * 1_{\Cup} \subseteq R \Cap \sim R$ and $R \odot 1_{\Cap} \subseteq R \Cup \sim R$.

The inner isomorphism tells us that the interaction of Peleg co-composition with the outer operations is as weak as that of Peleg composition. Peleg co-composition preserves $\cup$ in its first argument and $\subseteq$ in its second one. Moreover $R \odot (S \Cap T) \subseteq (R \odot S) \Cap (R \odot T)$ and $(R \Cup S) \odot T \subseteq (R \odot T) \Cap (S \odot T)$, and $(R \Cup S) \odot T = (R \odot T) \Cap (S \odot T)$ whenever $T \Cap T \subseteq T$.

Intersection-closure is defined analogously to union-closure with respect to the inner intersection $\bigCap_{i \in I} R_i$ of a family of multirelations R_i. The isomorphism $\sim$ extends from finite inner union and intersections to arbitrary ones. For intersection-closed T, we have $(\bigCup_{i \in I} R_i) \odot T = \bigCap_{i \in I} (R_i \odot T)$ for each I.

Intersection-closed multirelations are called "multiplicative" in [29, 30], noting distributivity properties of Parikh composition over intersections. Here we obtain distributivity results of Peleg (co-)composition over inner unions. The dual additivity property studied by [29, 30], however, differs from union-closure.

Down-closed multirelations are intersection-closed. Further, Proposition 4.6 implies that $R\uparrow = (\sim R) \odot (\sim 1)\uparrow$ by inner duality using Lemma 4.1. Note that $(\sim 1)\uparrow = \sim(1\downarrow) = (1_{\Cap} \cup \sim 1)$, so that $R\uparrow = (\sim R) \odot (1_{\Cap} \cup \sim 1)$. Thus $(R \odot S)\uparrow = R \odot S\uparrow$ by the inner isomorphism.

The interaction of Peleg co-composition with the inner preorders is weak: the operation $\odot$ preserves $\sqsubseteq_{\downarrow}$, $\sqsubseteq_{\uparrow}$, $\sqsubseteq_{\updownarrow}$, $=_{\downarrow}$, $=_{\uparrow}$ and $=_{\updownarrow}$ in its second argument. Furthermore, $R \sqsubseteq_{\uparrow} R \odot 1_{\Cap}$.

7. Conclusion

We have studied the inner structure of multirelations and their interaction with Peleg composition in the language of relation algebra and universal algebra. We have considered in particular the operations of inner and outer union, intersection and complementation, a duality between the inner and outer levels, up-closures and down-closures of multirelations and the associated preorders and equivalences, with a view on their structure and their use in future algebraic axiomatisations.

In the second article of this trilogy [10] we use the results obtained here to study inner and outer univalent and deterministic multirelations and their categories, and introduce determinisation maps from multirelations to inner and outer deterministic multirelations. Here, the focus shifts from universal algebra to power allegories and from quantales to quantaloids. In the third article [11] we use these maps to develop an algebraic approach to modal operators on multirelations, related to previous work by Nerode and Wijesekera [23] and Goldblatt [16].

Based on the multirelational language of concrete relations and multirelations and its properties in this work, an axiomatic extension of the relation algebra used in this article with multirelational operations is its most natural continuation. It also remains to consider other families of multirelations, in particular up-closed and convex-closed ones, and multiplications other than Peleg composition in relationship to the approach in this article, beyond the initial work by Rewitzky [29].

Acknowledgement. Hitoshi Furusawa and Walter Guttmann thank the Japan Society for the Promotion of Science for supporting part of this research through a JSPS Invitational Fellowship for Research in Japan.

Appendix A. Basis

Almost every operation in this article can be defined in terms of a basis of 6 operations that mix the relational and the multirelational language: the relational operations $-$, $\cap$, $/$ and the multirelational operations 1, $\Cup$, $*$:

- $R \cup S = -(-R \cap -S)$
- $R - S = R \cap -S$
- $\emptyset = R \cap -R$
- $U = -\emptyset$
- $R\!\uparrow\, = R \Cup U$
- $\in\, = 1\!\uparrow$
- $Id = 1/1$
- $R^\smile = -(-Id/R)$
- $SR = -(-S/R^\smile)$
- $R \backslash S = (S^\smile / R^\smile)^\smile$
- $R \div S = (R \backslash S) \cap (R^\smile / S^\smile)$
- $R_{\mathcal{P}} = \in R^\smile \in \div \in$
- $\Omega = \in \backslash \in$
- $C = \in \div - \in$
- $\sim\! R = RC$
- $R \Cap S = \sim\!(\sim\! R \Cup \sim\! S)$
- $R\!\downarrow\, = R \Cap U$
- $R\!\updownarrow\, = R\!\uparrow \cap R\!\downarrow$
- $1_{\Cup} = 1 \Cap \sim\! 1$
- $1_{\Cap} = \sim\! 1_{\Cup}$
- $R^{\mathsf{d}} = -\!\sim\! R$
- $R \odot S = \sim\!(R * \sim\! S)$
- $R_* = ((1^\smile \in \div \in) * 1^\smile R1) Id_{\mathcal{P}}$
- $\mathsf{A}_{\Cup} = U1$
- $\mathsf{A}_{\Cap} = \sim\! \mathsf{A}_{\Cup}$
- $dom(R) = Id \cap RR^\smile$
- $R \sqsubseteq_{\uparrow} S \Leftrightarrow S \subseteq R\!\uparrow$
- $R \sqsubseteq_{\downarrow} S \Leftrightarrow R \subseteq S\!\downarrow$
- $R \sqsubseteq_{\updownarrow} S \Leftrightarrow R \sqsubseteq_{\downarrow} S \wedge R \sqsubseteq_{\uparrow} S$

Since $Id : \mathcal{P}Y \nrightarrow \mathcal{P}Y$ is also a multirelation (the source of which happens to be a power set), the simpler definition $R_* = Id * R$ may be used. Alternatively, we could of course replace Peleg composition by Peleg lifting in the basis. Finally, relational / is required to define some of the operations in our list as it is the only operation in the basis that can change types. We have so far not attempted to axiomatise the basic operations in the sense of (heterogeneous) relation algebra [34], concurrent dynamic algebra [15] or likewise.

References

[1] Back, R.J., von Wright, J., 1998. Refinement Calculus. Springer, New York.

[2] Bellier, D., Benerecetti, M., Della Monica, D., Mogavero, F., 2023. Good-for-game QPTL: An alternating Hodges semantics. ACM Transactions on Computational Logic 24, 1–57.

[3] van Benthem, J., Ghosh, S., Liu, F., 2008. Modelling simultaneous games in dynamic logic. Synthese 165, 247–268.

[4] Berghammer, R., Guttmann, W., 2015. A relation-algebraic approach to multirelations and predicate transformers, in: Hinze, R., Voigtländer, J. (Eds.), Mathematics of Program Construction. Springer, volume 9129 of Lecture Notes in Computer Science, pp. 50–70.

[5] Berghammer, R., Guttmann, W., 2017. An algebraic approach to multirelations and their properties. Journal of Logical and Algebraic Methods in Programming 88, 45–63.

[6] Burris, S., Sankappanavar, H.P., 1981. A Course in Universal Algebra. Springer.

[7] Cavalcanti, A., Woodcock, J., Dunne, S., 2006. Angelic nondeterminism in the unifying theories of programming. Formal Aspects of Computing 18, 288–307.

[8] Davey, B.A., Priestley, H.A., 2002. Introduction to Lattices and Order. Second ed., Cambridge University Press.

[9] Freyd, P.J., Ščedrov, A., 1990. Categories, Allegories. volume 39 of North-Holland Mathematical Library. Elsevier Science Publishers.

[10] Furusawa, H., Guttmann, W., Struth, G., 2024. Determinism of multirelations. Journal of Logical and Algebraic Methods in Programming 139:100976.

[11] Furusawa, H., Guttmann, W., Struth, G., 2025. Modal algebra of multirelations. Journal of Logic and Computation 35. https://doi.org/10.1093/logcom/exae023.

[12] Furusawa, H., Kawahara, Y., Struth, G., Tsumagari, N., 2017. Kleisli, Parikh and Peleg compositions and liftings for multirelations. Journal of Logical and Algebraic Methods in Programming 90, 84–101.

[13] Furusawa, H., Struth, G., 2015a. Binary multirelations. Archive of Formal Proofs, Formal proof development, https://isa-afp.org/entries/Multirelations.html.

[14] Furusawa, H., Struth, G., 2015b. Concurrent dynamic algebra. ACM Transactions on Computational Logic 16, 1–38.

[15] Furusawa, H., Struth, G., 2016. Taming multirelations. ACM Transactions on Computational Logic 17, 1–34.

[16] Goldblatt, R., 1992. Parallel action: Concurrent dynamic logic with independent modalities. Studia Logica 51, 551–578.

[17] Grillet, P.A., 1971. Regular Categories. Lecture Notes in Mathematics, Springer.

[18] Guttmann, W., 2014. Multirelations with infinite computations. Journal of Logical and Algebraic Methods in Programming 83, 194–211.

[19] Guttmann, W., Struth, G., 2023. Inner structure, determinism and modal algebra of multirelations. Archive of Formal Proofs, Formal proof development, https://isa-afp.org/entries/Multirelations_Heterogeneous.html.

[20] Martin, C.E., Curtis, S.A., Rewitzky, I., 2007. Modelling angelic and demonic nondeterminism with multirelations. Sci. Comput. Program. 65, 140–158.

[21] McIver, A., Morgan, C., 2005. Abstraction, Refinement and Proof for Probabilistic Systems. Springer.

[22] McIver, A.K., Weber, T., 2005. Towards automated proof support for probabilistic distributed systems, in: Sutcliffe, G., Voronkov, A. (Eds.), Logic for Programming, Artificial Intelligence, and Reasoning. Springer, volume 3835 of Lecture Notes in Computer Science, pp. 534–548.

[23] Nerode, A., Wijesekera, D., 1990. Constructive Concurrent Dynamic Logic I. Technical Report Mathematical Sciences Institute 90-43. Cornell University.

[24] Olivier, J.P., Serrato, D., 1980. Catégories de Dedekind: Morphismes transitifs dans les catégories de Schröder. Comptes rendus hebdomadaires des séances de l'Académie des Sciences, Série A 290, 939–941.

[25] Parikh, R., 1983. Propositional game logic, in: 24th Annual Symposium on Foundations of Computer Science. IEEE Computer Society, pp. 195–200.

[26] Parikh, R., 1985. The logic of games and its applications, in: Karpinski, M., van Leeuwen, J. (Eds.), Topics in the Theory of Computation. North-Holland Publishing Company, volume 102 of North-Holland Mathematics Studies, pp. 111–139.

[27] Pauly, M., Parikh, R., 2003. Game logic – an overview. Studia Logica 75, 165–182.

[28] Peleg, D., 1987. Concurrent dynamic logic. J. ACM 34, 450–479.

[29] Rewitzky, I., 2003. Binary multirelations, in: de Swart, H., Orłowska, E., Schmidt, G., Roubens, M. (Eds.), Theory and Applications of Relational Structures as Knowledge Instruments. Springer, volume 2929 of Lecture Notes in Computer Science, pp. 256–271.

[30] Rewitzky, I., Brink, C., 2006. Monotone predicate transformers as up-closed multirelations, in: Schmidt, R.A. (Ed.), Relations and Kleene Algebra in Computer Science. Springer, volume 4136 of Lecture Notes in Computer Science, pp. 311–327.

[31] Rosenthal, K.I., 1990. Quantales and Their Applications. Longman Scientific and Technical.

[32] Rosenthal, K.I., 1996. The Theory of Quantaloids. Addison Wesley Longman Limited.

[33] Schmidt, G., 2011. Relational Mathematics. Cambridge University Press.

[34] Schmidt, G., Ströhlein, T., 1989. Relationen und Graphen. Springer.

[35] Smyth, M.B., 1983. Power domains and predicate transformers: A topological view, in: Díaz, J. (Ed.), Automata, Languages and Programming. Springer, volume 154 of Lecture Notes in Computer Science, pp. 662–675.

[36] Tarski, A., 1941. On the calculus of relations. The Journal of Symbolic Logic 6, 73–89.

[37] Tsumagari, N., 2012. Probability meets non-probability via complete IL-semirings. Ph.D. thesis. Kagoshima University.

[38] Weber, T., 2008. SAT-based Finite Model Generation for Higher-Order Logic. Ph.D. thesis. TU München.