Dependences between Domain Constructions in Heterogeneous Relation Algebras

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Abstract. We show the following dependences between relational domain constructions in the framework of heterogeneous relation algebras. If all power sets and subsets exist and objects are comparable, then all sums exist. If all sums exist and atoms are rectangular, then all products exist. If all atoms are rectangular, then all subsets exist if and only if all quotients exist. We give models with rectangular atoms which rule out further dependences between these constructions.

1 Introduction

Applications of relations often need to work with structured data. To facilitate this, extensions of relation algebras by various domain constructions have been studied in the literature [2–6, 11–14, 20]. Examples include power sets, products, sums, quotients and subsets: well-known basic ingredients for the construction of more complex data types.

A typical way to extend relation algebras by a domain construction is to introduce operations axiomatically and to prove that the axioms characterise the intended domain uniquely up to isomorphism. Hence it is natural to ask about the independence of the axioms used for the various domain constructions. Studying their dependences is the topic of this paper.

We work in the framework of heterogeneous relation algebras and contribute the following main results:

- If all power sets and subsets exist and objects are comparable, then all sums exist.
- If all sums exist and atoms are rectangular, then all products exist.
- If all atoms are rectangular, then all subsets exist if and only if all quotients exist.
- There are models with rectangular atoms which rule out further dependences between these constructions.

We first recall basic definitions and properties of heterogeneous relation algebras, and the domain constructions of power set, product, sum, quotient and subset. Sections 3–6 provide the dependence results. This is followed by models for the independence results in Section 7.

2 Heterogeneous Relation Algebras and Domain Constructions

In this section we define heterogeneous relation algebras and the domain constructions of power set, product, sum, quotient and subset. Heterogeneous relation algebras are a typed version of Tarski's relation algebras [15]. Related frameworks in which the dependence of domain constructions could be studied are allegories and Dedekind categories [4, 6, 10].

2.1 Heterogeneous Relation Algebras

The following definition is from [12] which provides a good overview of heterogeneous relation algebras and some domain constructions; also see [13].

Definition 1. A *heterogeneous relation algebra* is a locally small category with objects Obj, morphisms $\mathrm{Mor}(A, B)$ for $A, B \in \mathrm{Obj}$, composition ;, identities I_A, and the following additional structure.

- For each $A, B \in \mathrm{Obj}$ there is a transposition ${}^{\mathsf{T}}_{A,B} : \mathrm{Mor}(A, B) \to \mathrm{Mor}(B, A)$.
- Each $\mathrm{Mor}(A, B)$ is a complete atomic Boolean algebra with join $\sqcup_{A,B}$, meet $\sqcap_{A,B}$, complement $\overline{}_{A,B}$, order $\sqsubseteq_{A,B}$, least element $\mathsf{O}_{A,B}$ and greatest element $\mathsf{L}_{A,B}$, where $\mathsf{O}_{A,B} \neq \mathsf{L}_{A,B}$.
- Each $Q \in \mathrm{Mor}(A, B)$ and $R \in \mathrm{Mor}(B, C)$ and $S \in \mathrm{Mor}(A, C)$ satisfy the Schröder equivalences $Q \,;\, R \sqsubseteq_{A,C} S \Leftrightarrow Q^{\mathsf{T}} \,;\, \overline{S} \sqsubseteq_{B,C} \overline{R} \Leftrightarrow \overline{S} \,;\, R^{\mathsf{T}} \sqsubseteq_{A,B} \overline{Q}$.
- The Tarski rule $R \neq \mathsf{O}_{A,B} \Leftrightarrow \mathsf{L}_{C,A} \,;\, R \,;\, \mathsf{L}_{B,D} = \mathsf{L}_{C,D}$ holds for each $R \in \mathrm{Mor}(A, B)$ and $C, D \in \mathrm{Obj}$.

We usually omit subscripts specifying type information and abbreviate composition $R;S$ as RS. Morphisms $R \in \mathrm{Mor}(A, B)$ are called *relations* and denoted $R : A \leftrightarrow B$.

An example of a heterogenous relation algebra is REL, which has all non-empty sets as objects, all (set-theoretic) binary relations $R \subseteq A \times B$ as morphisms $R : A \leftrightarrow B$, and the usual operations on binary relations. Further examples appear throughout this paper.

Relation R is *univalent* if $R^{\mathsf{T}} R \sqsubseteq \mathsf{I}$, *total* if $\mathsf{I} \sqsubseteq RR^{\mathsf{T}}$, a *mapping* if R is univalent and total, and *injective/surjective/bijective* if R^{T} is a univalent/total/a mapping. Relation R is *reflexive* if $\mathsf{I} \sqsubseteq R$, *symmetric* if $R^{\mathsf{T}} = R$, *transitive* if $RR \sqsubseteq R$, a *partial equivalence* if R is symmetric and transitive, and an *equivalence* if R is reflexive and a partial equivalence. Relation R is a *partial identity* if $R \sqsubseteq \mathsf{I}$. Relation R is a *vector* if $R = R\mathsf{L}$ and *rectangular* if $R\mathsf{L}R \sqsubseteq R$.

Partial identities are symmetric and they form a Boolean algebra in which composition coincides with meet and complement is given by $\neg R = \overline{R\mathsf{L}} \sqcap \mathsf{I}$.

A number of residual operations will be useful in particular for the construction of power sets. The *left residual* of relations $Q : B \leftrightarrow A$ and $R : C \leftrightarrow A$ is $Q/R = \overline{\overline{Q}R^{\mathsf{T}}}$. The *right residual* of relations $Q : A \leftrightarrow B$ and $R : A \leftrightarrow C$

is $Q \backslash R = \overline{Q^{\mathsf{T}} \overline{R}}$. Their *symmetric quotient* is $Q \div R = (Q \backslash R) \sqcap (Q^{\mathsf{T}} / R^{\mathsf{T}})$. The following table summarises the logical interpretation of these operations in REL:

$$(x, y) \in Q/R \Leftrightarrow (\forall z \in A : (y, z) \in R \Rightarrow (x, z) \in Q)$$
$$(x, y) \in Q \backslash R \Leftrightarrow (\forall z \in A : (z, x) \in Q \Rightarrow (z, y) \in R)$$
$$(x, y) \in Q \div R \Leftrightarrow (\forall z \in A : (z, x) \in Q \Leftrightarrow (z, y) \in R)$$

The following lemma collects properties of the above operations used in this paper. Here we only prove Lemma 2.7. The other properties are known from the literature or simple consequences; in particular, see [11–13].

Lemma 2.

1. $QR \sqcap S \sqsubseteq (Q \sqcap SR^{\mathsf{T}})(R \sqcap Q^{\mathsf{T}} S)$.
2. $PQR \sqsubseteq S \Leftrightarrow P^{\mathsf{T}} \overline{S} R^{\mathsf{T}} \sqsubseteq \overline{Q}$.
3. $(Q\mathsf{L} \sqcap R)S = Q\mathsf{L} \sqcap RS$.
4. $(R \sqcap \mathsf{L}Q)S = R(Q^{\mathsf{T}}\mathsf{L} \sqcap S)$.
5. *R is total if and only if $R\mathsf{L} = \mathsf{L}$.*
6. *R is rectangular if and only if $R\mathsf{L}R = R$.*
7. *$R^{\mathsf{T}} R \neq \mathsf{O}$ if R is total.*
8. *$Q(R \sqcap S) = QR \sqcap QS$ if Q is univalent.*
9. *$R\mathsf{L} \sqcap \mathsf{I} = R = \mathsf{L}R \sqcap \mathsf{I}$ if R is a partial identity.*
10. $\mathsf{I} \backslash R = R = R/\mathsf{I}$.
11. *$\backslash$ reverses $\sqsubseteq$ in its first argument and preserves $\sqsubseteq$ in its second argument.*
12. $(Q \sqcup R) \backslash S = (Q \backslash S) \sqcap (R \backslash S)$.
13. *$Q(R \backslash S) = RQ^{\mathsf{T}} \backslash S$ if Q is a mapping.*
14. *$R \backslash QS \sqsubseteq Q^{\mathsf{T}} R \backslash S$ if Q is univalent.*
15. $QR \sqsubseteq S \Leftrightarrow Q \sqsubseteq S/R$.
16. $Q^{\mathsf{T}} / R^{\mathsf{T}} = (R \backslash Q)^{\mathsf{T}}$
17. $\mathsf{I} \div \mathsf{I} = \mathsf{I}$.
18. $\mathsf{I} \div \mathsf{O} = \mathsf{O}$.
19. $\mathsf{I} \sqsubseteq R \div R$.
20. $(Q \div R)^{\mathsf{T}} = R \div Q$.
21. $(Q \div R)(R \div S) = (Q \div S) \sqcap (Q \div R)\mathsf{L}$.
22. *$Q(R \div S) = RQ^{\mathsf{T}} \div S$ if Q is a mapping.*
23. *$(R \div S)Q = R \div SQ$ if Q is a bijective.*
24. *$Q^{\mathsf{T}} R \div S \sqsubseteq R \div QS$ if Q is injective and total.*
25. *$R \div Q$ is a vector if Q is a vector.*

Proof (of Lemma 2.7). Using Lemma 2.5, we have $R^{\mathsf{T}} R \sqsubseteq \mathsf{O} \Leftrightarrow R\mathsf{L} \sqsubseteq \overline{R} \Leftrightarrow \mathsf{L} \sqsubseteq \overline{R} \Leftrightarrow R \sqsubseteq \mathsf{O} \Leftrightarrow R\mathsf{L} \sqsubseteq \mathsf{O} \Leftrightarrow \mathsf{L} \sqsubseteq \mathsf{O}$, which is false. □

2.2 Power sets

Power sets are introduced in heterogeneous relation algebras by axiomatising the membership relation based on symmetric quotients [2]. The following axioms characterise power sets uniquely up to isomorphism; a similar remark holds for products, sums, quotients and subsets below.

Definition 3. The *power* of object A is an object 2^A with a relation $\varepsilon : A \leftrightarrow 2^A$ satisfying

- $\varepsilon \div \varepsilon \sqsubseteq \mathsf{I}$ and
- $R \div \varepsilon$ is total for each object B and relation $R : A \leftrightarrow B$.

It follows that 2^A is a categorical power object [4, 6]. In REL, 2^A is the usual power set of A and $(x, Y) \in \varepsilon \Leftrightarrow x \in Y$.

The following lemma collects properties of ε used in this paper. Here we only prove Lemmas 4.6–4.8. The other properties are known from the literature; in particular, see [8, 11, 12].

Lemma 4.

1. $\varepsilon \div \varepsilon = \mathsf{I}$.
2. $R \div \varepsilon$ *is a mapping.*
3. $\varepsilon(\varepsilon \div R) = R$.
4. $\varepsilon(\varepsilon \backslash R) = R$.
5. $(Q \div \varepsilon)(\varepsilon \div R) = Q \div R$.
6. $(\varepsilon \backslash \mathsf{I}) \div (\varepsilon \backslash \mathsf{I}) = \mathsf{I}$.
7. $\varepsilon \backslash \mathsf{I} = (\varepsilon \div \mathsf{O}) \sqcup (\varepsilon \div \mathsf{I})$.
8. $(\varepsilon \backslash \mathsf{I}) \div \mathsf{I} = \mathsf{O}$.

Proof (of Lemmas 4.6–4.8).

6. Using Lemmas 2.10, 2.11, 2.13, 2.19, 2.20, 4.2 and 4.3,
$$\begin{aligned}(\varepsilon\backslash\mathsf{I})\div(\varepsilon\backslash\mathsf{I}) &\sqsubseteq (\varepsilon\backslash\mathsf{I})\backslash(\varepsilon\backslash\mathsf{I}) \sqsubseteq (\varepsilon\div\mathsf{I})\backslash(\varepsilon\backslash\mathsf{I}) = (\mathsf{I}\div\varepsilon)(\mathsf{I}\backslash(\varepsilon\backslash\mathsf{I})) = (\mathsf{I}\div\varepsilon)(\varepsilon\backslash\mathsf{I}) \\ &= \varepsilon(\varepsilon\div\mathsf{I})\backslash\mathsf{I} = \mathsf{I}\backslash\mathsf{I} = \mathsf{I} \sqsubseteq (\varepsilon\backslash\mathsf{I})\div(\varepsilon\backslash\mathsf{I})\end{aligned}$$
7. $\varepsilon\div\mathsf{O} \sqsubseteq \varepsilon\backslash\mathsf{O} \sqsubseteq \varepsilon\backslash\mathsf{I}$ using Lemma 2.11 and $\varepsilon\div\mathsf{I} \sqsubseteq \varepsilon\backslash\mathsf{I}$. For the converse we have $(\varepsilon\backslash\mathsf{I}) \sqcap \overline{\varepsilon\div\mathsf{O}} = (\varepsilon\backslash\mathsf{I}) \sqcap \varepsilon^{\mathsf{T}}\mathsf{L} \sqsubseteq (\varepsilon\backslash\mathsf{I}) \sqcap \varepsilon^{\mathsf{T}}\varepsilon(\varepsilon\backslash\mathsf{I}) = (\varepsilon\backslash\mathsf{I}) \sqcap \varepsilon^{\mathsf{T}} = (\varepsilon\backslash\mathsf{I}) \sqcap (\varepsilon^{\mathsf{T}}/\mathsf{I}) = \varepsilon\div\mathsf{I}$ using Lemmas 2.1, 2.10 and 4.4. The result follows by shunting.
8. Using Lemmas 2.1, 2.10, 2.12, 2.13, 2.18, 2.20, 4.2, 4.5 and 4.7,
$$\begin{aligned}(\varepsilon\backslash\mathsf{I})\div\mathsf{I} &\sqsubseteq (\varepsilon\backslash\mathsf{I})\backslash\mathsf{I} = ((\varepsilon\div\mathsf{O}) \sqcup (\varepsilon\div\mathsf{I}))\backslash\mathsf{I} = ((\varepsilon\div\mathsf{O})\backslash\mathsf{I}) \sqcap ((\varepsilon\div\mathsf{I})\backslash\mathsf{I}) \\ &= (\mathsf{O}\div\varepsilon)(\mathsf{I}\backslash\mathsf{I}) \sqcap (\mathsf{I}\div\varepsilon)(\mathsf{I}\backslash\mathsf{I}) = (\mathsf{O}\div\varepsilon) \sqcap (\mathsf{I}\div\varepsilon) \sqsubseteq (\mathsf{I}\div\varepsilon)(\varepsilon\div\mathsf{O})(\mathsf{O}\div\varepsilon) \\ &= (\mathsf{I}\div\mathsf{O})(\mathsf{O}\div\varepsilon) = \mathsf{O}\end{aligned}$$ □

2.3 Products

Products are introduced in heterogeneous relation algebras by axiomatising their projections [13].

Definition 5. The *product* of objects A, B is an object $A \times B$ with relations $p_A : A \times B \leftrightarrow A$ and $p_B : A \times B \leftrightarrow B$ satisfying

- p_A and p_B are mappings,
- $p_A^{\mathsf{T}} p_B = \mathsf{L}$ and
- $p_A p_A^{\mathsf{T}} \sqcap p_B p_B^{\mathsf{T}} \sqsubseteq \mathsf{I}$.

It follows that p_A and p_B are surjective, $p_A^{\mathsf{T}} p_A = \mathsf{I}_A$ and $p_B^{\mathsf{T}} p_B = \mathsf{I}_B$ and $p_A p_A^{\mathsf{T}} \sqcap p_B p_B^{\mathsf{T}} = \mathsf{I}_{A \times B}$. In general, $A \times B$ is not a categorical product, but it is one in the wide subcategory of mappings in REL [4].

2.4 Sums

Sums model disjoint unions and are introduced in heterogeneous relation algebras by axiomatising their injections [5, 20].

Definition 6. The *sum* of objects A, B is an object $A + B$ with relations $i_A : A \leftrightarrow A \times B$ and $i_B : B \leftrightarrow A \times B$ satisfying

- i_A and i_B are injective mappings,
- $i_A i_B^\mathsf{T} = \mathsf{O}$ and
- $\mathsf{I} \sqsubseteq i_A^\mathsf{T} i_A \sqcup i_B^\mathsf{T} i_B$.

It follows that $i_A i_A^\mathsf{T} = \mathsf{I}_A$ and $i_B i_B^\mathsf{T} = \mathsf{I}_B$ and $i_A^\mathsf{T} i_A \sqcup i_B^\mathsf{T} i_B = \mathsf{I}_{A+B}$. Moreover $A + B$ is a categorical coproduct; in REL, it is also a categorical product [4].

2.5 Quotients

Quotients are based on equivalence relations and introduced in heterogeneous relation algebras by axiomatising the projection to equivalence classes [11].

Definition 7. The *quotient* of object A by equivalence $E : A \leftrightarrow A$ is an object A/E with a relation $p : A \leftrightarrow A/E$ satisfying

- $pp^\mathsf{T} = E$ and
- $p^\mathsf{T} p = \mathsf{I}$.

It follows that p is a surjective mapping. Hence A/E with p is a categorical quotient object [1].

2.6 Subsets

Subsets are based on partial identities and introduced in heterogeneous relation algebras by axiomatising the injection into the base set. We specialise the axioms of 'subset extrusion' given in [11] to subsets specified by a partial identity.

Definition 8. The *subset* of object A corresponding to non-zero partial identity $S : A \leftrightarrow A$ is an object S with a relation $i : S \leftrightarrow A$ satisfying

- $i^\mathsf{T} i = S$ and
- $ii^\mathsf{T} = \mathsf{I}$.

It follows that i is an injective mapping. Hence S with i is a categorical subobject [1]. We overload the name of the object S with the name of the partial identity S on which it is based, because the two closely correspond to each other.

We remark that the quotient and subset constructions are special cases of the construction of splittings [3, 6, 18]. Splittings are based on partial equivalence relations; they combine taking a subset to the domain of the partial equivalence and a quotient to its classes. Every partial identity is a partial equivalence relation. Since projections go from A to the quotient whereas injections go from the subset to A, one of the two directions has to be reversed if they are unified as splittings. In this paper, we study quotients and subsets separately; see Section 6 for their dependence.

3 Sums from Power Sets and Subsets

In this section we show that all sums exist if all power sets and subsets exist and objects are comparable. Note that both the power set construction and the subset construction are based on a single object, whereas sums are based on two objects. To combine two different objects we need a way to relate them. This is provided by the following concept.

Definition 9. Object A is *contained* in object B if there is an injective mapping $i : A \leftrightarrow B$. Objects A, B are *comparable* if A is contained in B or B is contained in A.

In REL, A is contained in B if and only if $|A| \leq |B|$, where $|\cdot|$ is the cardinality of a set, and any two objects are comparable (this is equivalent to the Axiom of Choice). For a heterogeneous relation algebra in which not all objects are comparable consider $\mathrm{Obj} = \{A, B\}$ where $A = \{1, 2\}$ and $B = \{1, 2, 3\}$ and $\mathrm{Mor}(A, A) = \{\mathsf{O}, \mathsf{I}, \overline{\mathsf{I}}, \mathsf{L}\}$ and $\mathrm{Mor}(B, B) = \{\mathsf{O}, \mathsf{I}, \overline{\mathsf{I}}, \mathsf{L}\}$ and $\mathrm{Mor}(A, B) = \{\mathsf{O}, \mathsf{L}\}$ and $\mathrm{Mor}(B, A) = \{\mathsf{O}, \mathsf{L}\}$. It is a (heterogeneous) subalgebra of REL but none of the morphisms between A and B are injective mappings: $\mathsf{O}\mathsf{O}^{\mathsf{T}} = \mathsf{O} \neq \mathsf{I}$ and $\mathsf{L}\mathsf{L}^{\mathsf{T}} = \mathsf{L} \neq \mathsf{I}$ for all well-typed instances of these (in)equations.

Containment is connected to the subset domain construction as the following result shows.

Lemma 10. *A is contained in B if and only if A is a subset of B corresponding to some non-zero partial identity S.*

Proof. For the forward implication, let $i : A \leftrightarrow B$ be the injective mapping arising from the containment. Define $S : B \leftrightarrow B$ by $S = i^{\mathsf{T}} i$. Then $S \sqsubseteq \mathsf{I}$ since i is univalent. Moreover $S \neq \mathsf{O}$ by Lemma 2.7 since i is total. Finally $ii^{\mathsf{T}} = \mathsf{I}$ is equivalent to i being injective and total. The backward implication follows immediately as the subset construction gives the desired injective mapping. □

The existence of all subsets does not imply that objects are comparable. This is shown by the above example, which contains all subsets since I is the only non-zero partial identity on each of the two objects. The converse implication also does not hold. The single-object relation algebra of all binary relations on a two-element set does not contain subsets corresponding to the two partial identities between O and I, but its single object is comparable with itself since I is injective.

After these preliminaries we turn to the main goal of constructing sums from power sets and subsets. The general idea is to represent a sum as the power set of a power set. To illustrate this, consider the set-theoretic example of constructing the disjoint union of $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Elements of A will be represented by singleton sets of singleton sets: 1 by $\{\{1\}\}$ and 2 by $\{\{2\}\}$ and 3 by $\{\{3\}\}$. Elements of B will be represented by including the empty set to distinguish the source: a by $\{\emptyset, \{a\}\}$ and b by $\{\emptyset, \{b\}\}$.

We implement this construction for general heterogeneous relation algebras in three parts. The first theorem will be used to discard sets of sets that are not used in the construction, but we formulate it more generally. This is where subsets come into play.

Theorem 11. *Assume all subsets exist. Assume $i_A : A \leftrightarrow C$ and $i_B : B \leftrightarrow C$ are injective mappings with $i_A i_B^\top = \mathsf{O}$. Then $A + B$ exists.*

Proof. Define $S : C \leftrightarrow C$ by $S = i_A^\top i_A \sqcup i_B^\top i_B$. Then $S \sqsubseteq \mathsf{I}$ since i_A and i_B are univalent. Moreover $S \neq \mathsf{O}$ by Lemma 2.7 since i_A is total. Hence the subset S exists with injection $i : S \leftrightarrow C$ satisfying $i^\top i = S$ and $ii^\top = \mathsf{I}$. We show $S = A + B$. To this end, define $j_A : A \leftrightarrow S$ by $j_A = i_A i^\top$ and $j_B : B \leftrightarrow S$ by $j_B = i_B i^\top$. Then

- $j_A j_A^\top = i_A i^\top i i_A^\top = i_A S i_A^\top = i_A i_A^\top i_A i_A^\top \sqcup i_A i_B^\top i_B i_A^\top = \mathsf{I} \sqcup \mathsf{O} = \mathsf{I}$ since i_A is injective and total.
- $j_B j_B^\top = i_B i^\top i i_B^\top = i_B S i_B^\top = i_B i_A^\top i_A i_B^\top \sqcup i_B i_B^\top i_B i_B^\top = \mathsf{O} \sqcup \mathsf{I} = \mathsf{I}$ since i_B is injective and total.
- $j_A j_B^\top = i_A i^\top i i_B^\top = i_A S i_B^\top \sqsubseteq i_A i_B^\top = \mathsf{O}$.
- $j_A^\top j_A \sqcup j_B^\top j_B = i i_A^\top i_A i^\top \sqcup i i_B^\top i_B i^\top = i S i^\top = i i^\top i i^\top = \mathsf{I}$. $\square$

The next corollary instantiates the previous theorem to the singleton-set construction outlined above. This is where power sets come into play. In REL, $\mathsf{I} \div \varepsilon$ relates an element with the singleton set containing it, that is, $(x, Y) \in \mathsf{I} \div \varepsilon \Leftrightarrow Y = \{x\}$. Hence $(\mathsf{I} \div \varepsilon)(\mathsf{I} \div \varepsilon)$ constructs the desired doubly singleton sets. Note that $(\mathsf{I} \div \varepsilon)(\mathsf{I} \div \varepsilon) = (\varepsilon \div \mathsf{I}) \div \varepsilon$ by Lemmas 2.20, 2.22 and 4.2. To include the empty set we allow subsets of the singleton set inside the outer set by replacing the inner symmetric quotient with a right residual as in $(\varepsilon \backslash \mathsf{I}) \div \varepsilon$.

Corollary 12. *Assume all subsets and power sets exist. Then $A + A$ exists for each object A.*

Proof. Define $i_A, i_B : A \leftrightarrow 2^{2^A}$ by $i_A = (\mathsf{I} \div \varepsilon)(\mathsf{I} \div \varepsilon)$ and $i_B = (\varepsilon \backslash \mathsf{I}) \div \varepsilon$. Then the assumptions of Theorem 11 are satisfied since

- $i_A i_A^\top = (\mathsf{I} \div \varepsilon)(\mathsf{I} \div \varepsilon)(\varepsilon \div \mathsf{I})(\varepsilon \div \mathsf{I}) = (\mathsf{I} \div \varepsilon)(\mathsf{I} \div \mathsf{I})(\varepsilon \div \mathsf{I}) = (\mathsf{I} \div \varepsilon)(\varepsilon \div \mathsf{I}) = \mathsf{I} \div \mathsf{I} = \mathsf{I}$ using Lemmas 2.17, 2.20 and 4.5.
- $i_B i_B^\top = ((\varepsilon \backslash \mathsf{I}) \div \varepsilon)(\varepsilon \div (\varepsilon \backslash \mathsf{I})) = (\varepsilon \backslash \mathsf{I}) \div (\varepsilon \backslash \mathsf{I}) = \mathsf{I}$ using Lemmas 2.20, 4.5 and 4.6.
- $i_A i_B^\top = (\mathsf{I} \div \varepsilon)(\mathsf{I} \div \varepsilon)(\varepsilon \div (\varepsilon \backslash \mathsf{I})) = (\mathsf{I} \div \varepsilon)(\mathsf{I} \div (\varepsilon \backslash \mathsf{I})) = \mathsf{O}$ using Lemmas 2.20, 4.5 and 4.8.
- $i_A^\top i_A = (\varepsilon \div \mathsf{I})(\varepsilon \div \mathsf{I})(\mathsf{I} \div \varepsilon)(\mathsf{I} \div \varepsilon) \sqsubseteq (\varepsilon \div \mathsf{I})(\varepsilon \div \varepsilon)(\mathsf{I} \div \varepsilon) = (\varepsilon \div \mathsf{I})(\mathsf{I} \div \varepsilon) \sqsubseteq \varepsilon \div \varepsilon = \mathsf{I}$ using Lemmas 2.20, 2.21 and 4.1.
- $i_B^\top i_B = (\varepsilon \div (\varepsilon \backslash \mathsf{I}))((\varepsilon \backslash \mathsf{I}) \div \varepsilon) \sqsubseteq \varepsilon \div \varepsilon = \mathsf{I}$ using Lemmas 2.20, 2.21 and 4.1. $\square$

The following corollary generalises this to sums of different objects. This is where comparability comes into play. The proof reuses calculations from the proof of Corollary 12, keeping i_A and modifying i_B by composing the injection available through comparability.

Corollary 13. *Assume all subsets and power sets exist and objects are comparable. Then $A + B$ exists for each object A, B.*

Proof. Without loss of generality assume B is contained in A using injection $i : B \leftrightarrow A$ (a symmetric argument applies in the other case). Define $i_A : A \leftrightarrow 2^{2^A}$ by $i_A = (\mathsf{I} \div \varepsilon)(\mathsf{I} \div \varepsilon)$ and $i_B : B \leftrightarrow 2^{2^A}$ by $i_B = i((\varepsilon \backslash \mathsf{I}) \div \varepsilon)$. Then the assumptions of Theorem 11 (not already covered by the proof of Corollary 12) are satisfied since

- $i_B i_B^\mathsf{T} = i i^\mathsf{T} = \mathsf{I}$ since i is injective and total.
- $i_A i_B^\mathsf{T} = \mathsf{O} i^\mathsf{T} = \mathsf{O}$.
- $i_B^\mathsf{T} i_B = (\varepsilon \div (\varepsilon \backslash \mathsf{I})) i^\mathsf{T} i ((\varepsilon \backslash \mathsf{I}) \div \varepsilon) \sqsubseteq (\varepsilon \div (\varepsilon \backslash \mathsf{I}))((\varepsilon \backslash \mathsf{I}) \div \varepsilon) \sqsubseteq \mathsf{I}$ as i is univalent. □

4 Products from Power Sets and Subsets

In this section we show that all products exist if all power sets and subsets exist and atoms are rectangular.

We recall concepts related to atoms; for example, see [7]. As usual, Q is an atom if $Q \neq \mathsf{O}$ and, for each $R \sqsubseteq Q$, either $R = Q$ or $R = \mathsf{O}$. Each $\mathrm{Mor}(A, B)$ is atomic, which means every $R \neq \mathsf{O}$ contains an atom $Q \sqsubseteq R$. Every atomic Boolean algebra is also atomistic, that is, every element is the supremum of the atoms below it. We denote the atomic partial identities of object A by $\mathrm{at}_1(A) = \{Q : A \leftrightarrow A \mid Q \text{ is an atom} \wedge Q \sqsubseteq \mathsf{I}\}$. Two atoms are either equal or their meet is O.

We remark about representability, since some of the following results assume that all atoms are rectangular. In a single-object relation algebra, this condition implies that the algebra is point-dense and therefore representable [9]. This consequence is not surprising as single-object relation algebras in which products exist are known to be representable by having conjugated quasi-projections [16].

The following result relates rectangular atoms to comparability.

Theorem 14. *Assume all atoms are rectangular. Then all objects are comparable.*

Proof. Let A, B be objects. Without loss of generality assume $|\mathrm{at}_1(A)| \leq |\mathrm{at}_1(B)|$ (otherwise swap A, B). Hence there is an injective function $g : \mathrm{at}_1(A) \to \mathrm{at}_1(B)$. Define $i : A \leftrightarrow B$ by $i = \bigsqcup_{a \in \mathrm{at}_1(A)} a \mathsf{L} g(a)$. Then

$$\begin{aligned}
i i^\mathsf{T} &= \bigsqcup_{a,b \in \mathrm{at}_1(A)} a \mathsf{L} g(a) g(b) \mathsf{L} b && \text{composition is completely distributive} \\
&= \bigsqcup_{a \in \mathrm{at}_1(A)} a \mathsf{L} g(a) \mathsf{L} a && g(a)g(b) \neq \mathsf{O} \Leftrightarrow g(a) = g(b) \Leftrightarrow a = b \\
&= \bigsqcup_{a \in \mathrm{at}_1(A)} a \mathsf{L} a && \text{Tarski rule} \\
&= \bigsqcup_{a \in \mathrm{at}_1(A)} a && \text{rectangular atoms, Lemma 2.6} \\
&= \mathsf{I} && \text{atomistic lattice}
\end{aligned}$$

$$
\begin{aligned}
i^{\mathsf{T}} i &= \bigsqcup_{a,b \in \mathrm{at}_1(A)} g(a)\mathsf{L}ab\mathsf{L}g(b) && \text{composition is completely distributive}\\
&= \bigsqcup_{a \in \mathrm{at}_1(A)} g(a)\mathsf{L}a\mathsf{L}g(a) && ab \neq \mathsf{O} \Leftrightarrow a = b\\
&= \bigsqcup_{a \in \mathrm{at}_1(A)} g(a)\mathsf{L}g(a) && \text{Tarski rule}\\
&= \bigsqcup_{a \in \mathrm{at}_1(A)} g(a) && \text{rectangular atoms, Lemma 2.6}\\
&\sqsubseteq \mathsf{I} && g(a) \text{ partial identity} \qquad \square
\end{aligned}
$$

The converse implication does not hold. Any non-representable single-object relation algebra is a counterexample.

After these preliminaries we turn to the main goal of constructing products from power sets, subsets and sums. The general idea is to represent a product as a power set of a sum. To illustrate this, consider the set-theoretic example of constructing the Cartesian product of $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Every pair will be represented by a two-element set: for example, $(3, a)$ by $\{3, a\}$. Even if $A = B$ the sum construction will tag the components so that the set representing a pair does not collapse to a singleton set.

We implement this construction for general heterogeneous relation algebras in three parts. The first theorem will be used to discard sets that are not used in the construction, but we formulate it more generally. This is where subsets come into play.

Theorem 15. *Assume all subsets exist. Assume $p_A : C \leftrightarrow A$ and $p_B : C \leftrightarrow B$ are univalent with $p_A^{\mathsf{T}} p_B = \mathsf{L}$ and $p_A p_A^{\mathsf{T}} \sqcap p_B p_B^{\mathsf{T}} \sqsubseteq \mathsf{I}$. Then $A \times B$ exists.*

Proof. Define $S : C \leftrightarrow C$ by $S = p_A \mathsf{L} p_B^{\mathsf{T}} \sqcap \mathsf{I}$. Then $S \sqsubseteq \mathsf{I}$ and $S \neq \mathsf{O}$ since otherwise $p_A \mathsf{L} p_B^{\mathsf{T}} \sqsubseteq \overline{\mathsf{I}}$, which is equivalent to $p_A^{\mathsf{T}} \mathsf{I} p_B \sqsubseteq \mathsf{O}$ using Lemma 2.2, whence $\mathsf{L} \sqsubseteq \mathsf{O}$. Hence the subset S exists with injection $i : S \leftrightarrow C$ satisfying $i^{\mathsf{T}} i = S$ and $i i^{\mathsf{T}} = \mathsf{I}$. We show $S = A \times B$. To this end, define $q_A : S \leftrightarrow A$ by $q_A = i p_A$ and $q_B : S \leftrightarrow B$ by $q_B = i p_B$. Then

- $q_A^{\mathsf{T}} q_A = p_A^{\mathsf{T}} i^{\mathsf{T}} i p_A = p_A^{\mathsf{T}} S p_A \sqsubseteq p_A^{\mathsf{T}} p_A \sqsubseteq \mathsf{I}$ since p_A is univalent.
- $q_B^{\mathsf{T}} q_B = p_B^{\mathsf{T}} i^{\mathsf{T}} i p_B = p_B^{\mathsf{T}} S p_B \sqsubseteq p_B^{\mathsf{T}} p_B \sqsubseteq \mathsf{I}$ since p_B is univalent.
- $\mathsf{I} = i i^{\mathsf{T}} i i^{\mathsf{T}} = i S i^{\mathsf{T}} \sqsubseteq i p_A (\mathsf{L} p_B^{\mathsf{T}} \sqcap p_A^{\mathsf{T}} \mathsf{I}) i^{\mathsf{T}} \sqsubseteq i p_A p_A^{\mathsf{T}} i^{\mathsf{T}} = q_A q_A^{\mathsf{T}}$ using Lemma 2.1.
- $\mathsf{I} = i S i^{\mathsf{T}} \sqsubseteq i (p_A \mathsf{L} \sqcap \mathsf{I} p_B) p_B^{\mathsf{T}} i^{\mathsf{T}} \sqsubseteq i p_B p_B^{\mathsf{T}} i^{\mathsf{T}} = q_B q_B^{\mathsf{T}}$ using Lemma 2.1.
- $q_A^{\mathsf{T}} q_B = p_A^{\mathsf{T}} i^{\mathsf{T}} i p_B = p_A^{\mathsf{T}} S p_B = p_A^{\mathsf{T}} (p_A \mathsf{L} \sqcap \mathsf{L} p_B^{\mathsf{T}} \sqcap \mathsf{I}) p_B = (p_A^{\mathsf{T}} \sqcap \mathsf{L} p_A^{\mathsf{T}})(p_B \sqcap p_B \mathsf{L}) = p_A^{\mathsf{T}} p_B = \mathsf{L}$ using Lemmas 2.3 and 2.4.
- $q_A q_A^{\mathsf{T}} \sqcap q_B q_B^{\mathsf{T}} = i p_A p_A^{\mathsf{T}} i^{\mathsf{T}} \sqcap i p_B p_B^{\mathsf{T}} i^{\mathsf{T}} = i (p_A p_A^{\mathsf{T}} \sqcap p_B p_B^{\mathsf{T}}) i^{\mathsf{T}} \sqsubseteq i i^{\mathsf{T}} = \mathsf{I}$ using Lemma 2.8 since i is univalent. $\square$

The next lemma carries out the two-element-set construction outlined above and establishes some of the properties of products. This is where power sets and sums come into play. The sets used in the construction contain a single element from A and a single element from B. In REL, $\mathsf{I} \div i_A \varepsilon$ relates an element of A with the sets over $A+B$ containing that element and arbitrary elements of B. Similarly, $\mathsf{I}{\div}i_B\varepsilon$ relates an element of B with sets over $A+B$ containing it and arbitrary elements of A. Hence the intermediate sets in the composition $(\mathsf{I}{\div}i_A\varepsilon)(i_B\varepsilon{\div}\mathsf{I})$ contain exactly one element of A and exactly one element of B.

Lemma 16. *Assume all power sets and sums exist. Let A,B be objects. Then there are an object C and univalent surjective relations $p_A : C \leftrightarrow A$ and $p_B : C \leftrightarrow B$ with $p_A p_A^\mathsf{T} \sqcap p_B p_B^\mathsf{T} \sqsubseteq \mathsf{I}$.*

Proof. Define $p_A : 2^{A+B} \leftrightarrow A$ by $p_A = i_A\varepsilon{\div}\mathsf{I}$ and $p_B : 2^{A+B} \leftrightarrow B$ by $p_B = i_B\varepsilon{\div}\mathsf{I}$. Then

- $p_A^\mathsf{T} p_A = (\mathsf{I}{\div}i_A\varepsilon)(i_A\varepsilon{\div}\mathsf{I}) = (\mathsf{I}{\div}\mathsf{I}) \sqcap (\mathsf{I}{\div}i_A\varepsilon)\mathsf{L} = \mathsf{I} \sqcap \mathsf{L} = \mathsf{I}$ using Lemmas 2.17, 2.20 and 2.21 and that $\mathsf{L} = (i_A^\mathsf{T}{\div}\varepsilon)\mathsf{L} \sqsubseteq (\mathsf{I}{\div}i_A\varepsilon)\mathsf{L}$ using Lemma 2.24.
- Similarly, $p_B^\mathsf{T} p_B = \mathsf{I}$.
- Finally,

$$\begin{aligned}
p_A p_A^\mathsf{T} \sqcap p_B p_B^\mathsf{T} &= (i_A\varepsilon{\div}\mathsf{I})(\mathsf{I}{\div}i_A\varepsilon) \sqcap (i_B\varepsilon{\div}\mathsf{I})(\mathsf{I}{\div}i_B\varepsilon) \\
&\sqsubseteq (i_A\varepsilon{\div}i_A\varepsilon) \sqcap (i_B\varepsilon{\div}i_B\varepsilon) \\
&= (i_A\varepsilon\backslash i_A\varepsilon) \sqcap (i_A\varepsilon\backslash i_A\varepsilon)^\mathsf{T} \sqcap (i_B\varepsilon\backslash i_B\varepsilon) \sqcap (i_B\varepsilon\backslash i_B\varepsilon)^\mathsf{T} \\
&\sqsubseteq (i_A^\mathsf{T} i_A\varepsilon\backslash\varepsilon) \sqcap (i_A^\mathsf{T} i_A\varepsilon\backslash\varepsilon)^\mathsf{T} \sqcap (i_B^\mathsf{T} i_B\varepsilon\backslash\varepsilon) \sqcap (i_B^\mathsf{T} i_B\varepsilon\backslash\varepsilon)^\mathsf{T} \\
&= ((i_A^\mathsf{T} i_A\varepsilon \sqcup i_B^\mathsf{T} i_B\varepsilon)\backslash\varepsilon) \sqcap ((i_A^\mathsf{T} i_A\varepsilon \sqcup i_B^\mathsf{T} i_B\varepsilon)\backslash\varepsilon)^\mathsf{T} \\
&= (\varepsilon\backslash\varepsilon) \sqcap (\varepsilon\backslash\varepsilon)^\mathsf{T} = \varepsilon{\div}\varepsilon = \mathsf{I}
\end{aligned}$$

  using Lemmas 2.12, 2.14, 2.16, 2.20, 2.21 and 4.1. □

The following corollary completes the assumptions of Theorem 15 by establishing $p_A^\mathsf{T} p_B = \mathsf{L}$. This is where rectangular atoms come into play.

Corollary 17. *Assume all subsets and power sets exist and atoms are rectangular. Then $A \times B$ exists for each object A, B.*

Proof. By Theorem 14 all objects are comparable. Hence by Corollary 13 all sums exist. Hence by Theorem 15 and the proof of Lemma 16, it remains to show $p_A^\mathsf{T} p_B = \mathsf{L}$ reusing $p_A = i_A\varepsilon{\div}\mathsf{I}$ and $p_B = i_B\varepsilon{\div}\mathsf{I}$. Since

$$\mathsf{L} = \mathsf{ILI} = (\bigsqcup_{a\in\mathrm{at}_1(A)} a)\,\mathsf{L}\,(\bigsqcup_{b\in\mathrm{at}_1(B)} b) = \bigsqcup_{\substack{a\in\mathrm{at}_1(A)\\ b\in\mathrm{at}_1(B)}} a\mathsf{L}b$$

it suffices to show $a\mathsf{L}b \sqsubseteq p_A^\mathsf{T} p_B$ for each $a \in \mathrm{at}_1(A)$ and $b \in \mathrm{at}_1(B)$. Consider such a and b, and let $v = i_A^\mathsf{T} a\mathsf{L} \sqcup i_B^\mathsf{T} b\mathsf{L}$. Since a is rectangular, $a\mathsf{L}a \sqsubseteq a \sqsubseteq \mathsf{I}$, whence $a \sqsubseteq \mathsf{I}/\mathsf{L}a$ using Lemma 2.15. Therefore

$$\begin{aligned}
a &\sqsubseteq a\mathsf{L} \sqcap (\mathsf{I}/\mathsf{L}a) = (\mathsf{I}\backslash a\mathsf{L}) \sqcap (\mathsf{I}/\mathsf{L}a) = \mathsf{I}{\div}a\mathsf{L} = \mathsf{I}{\div}i_A v \\
&= \mathsf{I}{\div}i_A\varepsilon(\varepsilon{\div}v) = (\mathsf{I}{\div}i_A\varepsilon)(\varepsilon{\div}v) = p_A^\mathsf{T}(\varepsilon{\div}v)
\end{aligned}$$

using Lemmas 2.10, 2.20, 2.23, 4.2 and 4.3. Similarly, $b \sqsubseteq p_B^\mathsf{T}(\varepsilon \div v)$. Hence

$$a\mathsf{L}b = a\mathsf{L}b^\mathsf{T} \sqsubseteq p_A^\mathsf{T}(\varepsilon \div v)\mathsf{L}(v \div \varepsilon)p_B = p_A^\mathsf{T}(\varepsilon \div v)(v \div \varepsilon)p_B \sqsubseteq p_A^\mathsf{T}(\varepsilon \div \varepsilon)p_B = p_A^\mathsf{T}p_B$$

using Lemmas 2.20, 2.21, 2.25 and 4.1 since v is a vector. $\square$

A referee noted that the construction of Lemma 16 was done in [17] and mentioned the following alternative to Corollary 17. If a heterogeneous relation algebra has powers, sums and products, then the product of A and B is isomorphic to the subset of 2^{A+B} in the above construction. Hence, if a heterogeneous relation algebra has powers and sums and is representable, then it can be embedded into REL, which has products, and the subset of 2^{A+B} is a product of A and B.

5 Products from Sums

In this section we show that all products exist if all sums exist and atoms are rectangular. This provides an alternative way to establish Corollary 17, but the proof in this section uses different ideas. We consider two cases, depending on whether $\mathrm{at}_1(B)$ is finite or infinite. In the finite case we represent $A \times B$ by an iterated sum $A + A + \cdots + A$ with as many summands as there are elements in $\mathrm{at}_1(B)$. In the infinite case we use that the cardinality of $\mathrm{at}_1(A) \times \mathrm{at}_1(B)$ is the same as the cardinality of $\mathrm{at}_1(A)$ or $\mathrm{at}_1(B)$ to obtain a bijection, so A or B will be the product.

Theorem 18. *Assume all sums exist and atoms are rectangular. Then $A \times B$ exists for each object A, B.*

Proof (if $\mathrm{at}_1(B)$ is finite). Let $b_1, \dots, b_n$ be the atomic partial identities of B. Define objects $A_1, \dots, A_n$ by $A_1 = A$ and $A_k = A_{k-1} + A$ with injections $i_k : A_{k-1} \leftrightarrow A_k$ and $j_k : A \leftrightarrow A_k$, for $2 \le k \le n$. Moreover, let $j_1 = \mathsf{I}$. We show $A_n = A \times B$.

Below, $i_{x..y}$ denotes the composition $i_x i_{x+1} \dots i_{y-1} i_y$ for indices $x \le y$; we also admit $i_{y+1..y} = \mathsf{I}$. The transposition is denoted $i_{y..x}^\mathsf{T} = i_y^\mathsf{T} i_{y-1}^\mathsf{T} \dots i_{x+1}^\mathsf{T} i_x^\mathsf{T}$.

Let $p_k : A_n \leftrightarrow A$ with $p_k = (j_k i_{k+1..n})^\mathsf{T}$ for $1 \le k \le n$. Define $p_A : A_n \leftrightarrow A$ by $p_A = \bigsqcup_{1 \le k \le n} p_k$ and $p_B : A_n \leftrightarrow B$ by $p_B = \bigsqcup_{1 \le k \le n} p_k \mathsf{L} b_k$.

We first show that $p_k^\mathsf{T} p_l$ is I if $k = l$ and O otherwise.

- If $k < l$, then $p_k^\mathsf{T} p_l = j_k i_{k+1..n} i_{n..l+1}^\mathsf{T} j_l^\mathsf{T} = j_k i_{k+1..l} j_l^\mathsf{T} = \mathsf{O}$.
- If $k > l$, then $p_k^\mathsf{T} p_l = (p_l^\mathsf{T} p_k)^\mathsf{T} = \mathsf{O}^\mathsf{T} = \mathsf{O}$.
- If $k = l$, then $p_k^\mathsf{T} p_l = j_k i_{k+1..n} i_{n..k+1}^\mathsf{T} j_k^\mathsf{T} = j_k j_k^\mathsf{T} = \mathsf{I}$.

The product axioms follow by

- $p_A^\mathsf{T} p_A = \bigsqcup\limits_{1 \le k,l \le n} p_k^\mathsf{T} p_l = \bigsqcup\limits_{1 \le k \le n} p_k^\mathsf{T} p_k = \mathsf{I}$.

– $p_B^{\mathsf{T}} p_B = \bigsqcup_{1\le k,l\le n} b_k^{\mathsf{T}} \mathsf{L} p_k^{\mathsf{T}} p_l \mathsf{L} b_l = \bigsqcup_{1\le k\le n} b_k \mathsf{L} p_k^{\mathsf{T}} p_k \mathsf{L} b_k = \bigsqcup_{1\le k\le n} b_k \mathsf{L} b_k = \bigsqcup_{1\le k\le n} b_k = \mathsf{I}$ since b_k is rectangular.

– $p_A^{\mathsf{T}} p_B = \bigsqcup_{1\le k,l\le n} p_k^{\mathsf{T}} p_l \mathsf{L} b_l = \bigsqcup_{1\le k\le n} p_k^{\mathsf{T}} p_k \mathsf{L} b_k = \bigsqcup_{1\le k\le n} \mathsf{L} b_k = \mathsf{L} \bigsqcup_{1\le k\le n} b_k = \mathsf{L}$.

– Finally,

$$
\begin{aligned}
p_A p_A^{\mathsf{T}} \sqcap p_B p_B^{\mathsf{T}} &= (\bigsqcup_{1\le k,l\le n} p_k p_l^{\mathsf{T}}) \sqcap (\bigsqcup_{1\le k,l\le n} p_k \mathsf{L} b_k b_l \mathsf{L} p_l^{\mathsf{T}}) \\
&= (\bigsqcup_{1\le k,l\le n} p_k p_l^{\mathsf{T}}) \sqcap (\bigsqcup_{1\le k\le n} p_k \mathsf{L} b_k \mathsf{L} p_k^{\mathsf{T}}) && b_k b_l \ne \mathsf{O} \Leftrightarrow b_k = b_l \\
&= (\bigsqcup_{1\le k,l\le n} p_k p_l^{\mathsf{T}}) \sqcap (\bigsqcup_{1\le k\le n} p_k \mathsf{L} p_k^{\mathsf{T}}) && \text{Tarski rule} \\
&= \bigsqcup_{1\le k,l,m\le n} p_k p_l^{\mathsf{T}} \sqcap p_m \mathsf{L} p_m^{\mathsf{T}} = \bigsqcup_{1\le k\le n} p_k p_k^{\mathsf{T}} = \mathsf{I} && \text{see below}
\end{aligned}
$$

For the second last equality we have $p_k p_l^{\mathsf{T}} \sqcap p_m \mathsf{L} p_m^{\mathsf{T}} \sqsubseteq p_k \mathsf{L} \sqcap p_m \mathsf{L} \sqsubseteq p_k p_k^{\mathsf{T}} p_m \mathsf{L} = \mathsf{O}$ using Lemma 2.1 if $k \ne m$. Similarly, $p_k p_l^{\mathsf{T}} \sqcap p_m \mathsf{L} p_m^{\mathsf{T}} \sqsubseteq \mathsf{L} p_l^{\mathsf{T}} \sqcap \mathsf{L} p_m^{\mathsf{T}} \sqsubseteq \mathsf{L} p_l^{\mathsf{T}} p_m p_m^{\mathsf{T}} = \mathsf{O}$ using Lemma 2.1 if $l \ne m$. Hence it suffices to take the join over indices $k = l = m$. The last equality is a consequence of $\bigsqcup_{1\le k\le l} p_k p_k^{\mathsf{T}} = i_{n..l+1}^{\mathsf{T}} i_{l+1..n}$, which we show by induction over l. The base case holds since $p_1 p_1^{\mathsf{T}} = i_{n..2}^{\mathsf{T}} j_1^{\mathsf{T}} j_1 i_{2..n} = i_{n..2}^{\mathsf{T}} i_{2..n}$ since $j_1 = \mathsf{I}$. The inductive case holds since

$$
\begin{aligned}
\bigsqcup_{1\le k\le l+1} p_k p_k^{\mathsf{T}} = \bigsqcup_{1\le k\le l} p_k p_k^{\mathsf{T}} \sqcup p_{l+1} p_{l+1}^{\mathsf{T}} &= i_{n..l+1}^{\mathsf{T}} i_{l+1..n} \sqcup i_{n..l+2}^{\mathsf{T}} j_{l+1}^{\mathsf{T}} j_{l+1} i_{l+2..n} \\
&= i_{n..l+2}^{\mathsf{T}} (i_{l+1}^{\mathsf{T}} i_{l+1} \sqcup j_{l+1}^{\mathsf{T}} j_{l+1}) i_{l+2..n} = i_{n..l+2}^{\mathsf{T}} i_{l+2..n}
\end{aligned}
$$

□

Proof (of Theorem 18 if $\mathrm{at}_1(B)$ *is infinite).* Let C be the 'bigger' of A and B; formally, let $C = A$ if $|\mathrm{at}_1(A)| \ge |\mathrm{at}_1(B)|$ and $C = B$ otherwise. We show $C = A \times B$.

We have $|\mathrm{at}_1(C)| \le |\mathrm{at}_1(A)| \cdot |\mathrm{at}_1(B)| = |\mathrm{at}_1(A) \times \mathrm{at}_1(B)|$. Conversely, $|\mathrm{at}_1(A)| \cdot |\mathrm{at}_1(B)| \le |\mathrm{at}_1(C)|^2 = |\mathrm{at}_1(C)|$ since $\mathrm{at}_1(C)$ is infinite. By the Cantor-Schröder-Bernstein theorem, there is a bijective function $g : \mathrm{at}_1(C) \to \mathrm{at}_1(A) \times \mathrm{at}_1(B)$. Define $p_A : C \leftrightarrow A$ and $p_B : C \leftrightarrow B$ by

$$
p_A = \bigsqcup_{\substack{a\in \mathrm{at}_1(C)\\ g(a)=(b,c)}} a \mathsf{L} b \qquad\qquad p_B = \bigsqcup_{\substack{a\in \mathrm{at}_1(C)\\ g(a)=(b,c)}} a \mathsf{L} c
$$

Then

– $p_A^{\mathsf{T}} p_A = \bigsqcup_{\substack{a,d\in \mathrm{at}_1(C)\\ g(a)=(b,c)\\ g(d)=(e,f)}} b \mathsf{L} a d \mathsf{L} e = \bigsqcup_{\substack{a\in \mathrm{at}_1(C)\\ g(a)=(b,c)}} b \mathsf{L} a \mathsf{L} b \sqsubseteq \bigsqcup_{\substack{a\in \mathrm{at}_1(C)\\ g(a)=(b,c)}} b \mathsf{L} b = \bigsqcup_{\substack{a\in \mathrm{at}_1(C)\\ g(a)=(b,c)}} b \sqsubseteq \mathsf{I}$ since b is rectangular.

– $p_B^{\mathsf{T}} p_B = \bigsqcup_{\substack{a,d\in \mathrm{at}_1(C)\\ g(a)=(b,c)\\ g(d)=(e,f)}} c \mathsf{L} a d \mathsf{L} f = \bigsqcup_{\substack{a\in \mathrm{at}_1(C)\\ g(a)=(b,c)}} c \mathsf{L} a \mathsf{L} c \sqsubseteq \bigsqcup_{\substack{a\in \mathrm{at}_1(C)\\ g(a)=(b,c)}} c \mathsf{L} c = \bigsqcup_{\substack{a\in \mathrm{at}_1(C)\\ g(a)=(b,c)}} c \sqsubseteq \mathsf{I}$ since c is rectangular.

– $p_A^{\mathsf{T}} p_B = \bigsqcup_{\substack{a,d \in \mathrm{at}_1(C)\\ g(a)=(b,c)\\ g(d)=(e,f)}} b\mathsf{L}ad\mathsf{L}f = \bigsqcup_{\substack{a \in \mathrm{at}_1(C)\\ g(a)=(b,c)}} b\mathsf{L}a\mathsf{L}c = \bigsqcup_{\substack{a \in \mathrm{at}_1(C)\\ g(a)=(b,c)}} b\mathsf{L}c = \bigsqcup_{\substack{b \in \mathrm{at}_1(A)\\ c \in \mathrm{at}_1(B)}} b\mathsf{L}c = (\bigsqcup_{b \in \mathrm{at}_1(A)} b)\,\mathsf{L}\,(\bigsqcup_{c \in \mathrm{at}_1(B)} c) = \mathsf{I}\mathsf{L}\mathsf{I} = \mathsf{L}$
using the Tarski rule and that g is bijective.

– Finally,

$$\begin{aligned}
p_A p_A^{\mathsf{T}} \sqcap p_B p_B^{\mathsf{T}} &= (\bigsqcup_{\substack{a,d \in \mathrm{at}_1(C)\\ g(a)=(b,c)\\ g(d)=(e,f)}} a\mathsf{L}be\mathsf{L}d) \sqcap (\bigsqcup_{\substack{a,d \in \mathrm{at}_1(C)\\ g(a)=(b,c)\\ g(d)=(e,f)}} a\mathsf{L}cf\mathsf{L}d) = (\bigsqcup_{\substack{a,d \in \mathrm{at}_1(C)\\ g(a)=(b,c)\\ g(d)=(b,f)}} a\mathsf{L}b\mathsf{L}d) \sqcap (\bigsqcup_{\substack{a,d \in \mathrm{at}_1(C)\\ g(a)=(b,c)\\ g(d)=(e,c)}} a\mathsf{L}c\mathsf{L}d) \\
&= (\bigsqcup_{\substack{a,d \in \mathrm{at}_1(C)\\ g(a)=(b,c)\\ g(d)=(b,f)}} a\mathsf{L}d) \sqcap (\bigsqcup_{\substack{a,d \in \mathrm{at}_1(C)\\ g(a)=(b,c)\\ g(d)=(e,c)}} a\mathsf{L}d) \\
&= \bigsqcup_{\substack{a,d,a',d' \in \mathrm{at}_1(C)\\ g(a)=(b,c),g(a')=(b',c')\\ g(d)=(b,f),g(d')=(e',c')}} a\mathsf{L}d \sqcap a'\mathsf{L}d' = \bigsqcup_{\substack{a,d \in \mathrm{at}_1(C)\\ g(a)=g(d)}} a\mathsf{L}d = \bigsqcup_{a \in \mathrm{at}_1(C)} a\mathsf{L}a = \bigsqcup_{a \in \mathrm{at}_1(C)} a = \mathsf{I}
\end{aligned}$$

using the Tarski rule, that g is bijective, that a is rectangular and Lemma 2.6. For the fourth last equality note that $a\mathsf{L}d \sqcap a'\mathsf{L}d' \sqsubseteq aa^{\mathsf{T}}a'\mathsf{L}d' = aa'\mathsf{L}d' = \mathsf{O}$ using Lemma 2.1 if $a \neq a'$. Similarly $a\mathsf{L}d \sqcap a'\mathsf{L}d' \sqsubseteq a\mathsf{L}dd'^{\mathsf{T}}d' = a\mathsf{L}dd' = \mathsf{O}$ using Lemma 2.1 if $d \neq d'$. Hence it suffices to take the join over indices $a = a'$ and $d = d'$. Moreover $g(a) = g(d)$ since $g(a)$ and $g(d)$ agree in their first components and $g(a')$ and $g(d')$ agree in their second components. □

6 Subsets from Quotients and Vice Versa

In this section we show that all subsets exist if and only if all quotients exist, if all atoms are rectangular. We start with a lemma about atoms.

Lemma 19. *Assume all atoms are rectangular. Let A, B be objects and let $a \in \mathrm{at}_1(A)$ and $b \in \mathrm{at}_1(B)$. Then $a\mathsf{L}b$ is an atom.*

Proof. First, $a\mathsf{L}b \neq \mathsf{O}$. Otherwise, $a \sqsubseteq a\mathsf{L} = a\mathsf{L}b\mathsf{L} = \mathsf{O}\mathsf{L} = \mathsf{O}$ using the Tarski rule, which would contradict that a is an atom. Hence there is an atom $c \sqsubseteq a\mathsf{L}b$. Then $c\mathsf{L} \sqcap \mathsf{I} \neq \mathsf{O}$ since otherwise $c\mathsf{L} \sqsubseteq \overline{\mathsf{I}}$, which is equivalent to $c^{\mathsf{T}} \sqsubseteq \mathsf{O}$, whence $c = \mathsf{O}$. Similarly $\mathsf{L}c \sqcap \mathsf{I} \neq \mathsf{O}$. Moreover $c\mathsf{L} \sqcap \mathsf{I} \sqsubseteq a\mathsf{L}b\mathsf{L} \sqcap \mathsf{I} \sqsubseteq a\mathsf{L} \sqcap \mathsf{I} = a$ and $\mathsf{L}c \sqcap \mathsf{I} \sqsubseteq \mathsf{L}a\mathsf{L}b \sqcap \mathsf{I} \sqsubseteq \mathsf{L}b \sqcap \mathsf{I} = b$ using Lemma 2.9. Hence $c\mathsf{L} \sqcap \mathsf{I} = a$ and $\mathsf{L}c \sqcap \mathsf{I} = b$ since a, b are atoms. Thus $a\mathsf{L}b = (c\mathsf{L} \sqcap \mathsf{I})\mathsf{L}(\mathsf{L}c \sqcap \mathsf{I}) \sqsubseteq c\mathsf{L}\mathsf{L}\mathsf{L}c \sqsubseteq c\mathsf{L}c \sqsubseteq c$ since c is rectangular. It follows that $a\mathsf{L}b = c$ is an atom. □

Theorem 20. *Assume all atoms are rectangular. Then all quotients exist if and only if all subsets exist.*

Proof (of forward implication). Let $S : A \leftrightarrow A$ be a non-zero partial identity. Hence there is an atom $a \sqsubseteq S$. Define $E : A \leftrightarrow A$ by $E = S \sqcup a\mathsf{L}\neg S \sqcup \neg S\mathsf{L}a \sqcup \neg S\mathsf{L}\neg S$. Then E is an equivalence:

- $\mathsf{I} = S \sqcup \neg S \sqsubseteq S \sqcup \neg S \mathsf{L} \neg S \sqsubseteq E$.
- $E^\mathsf{T} = E$ since S, $\neg S$ and a are partial identities and hence symmetric.
- Since $aS = Sa = a$ and $a\neg S = \neg Sa \sqsubseteq \neg SS = \mathsf{O}$ and a is rectangular, we obtain

$$\begin{aligned} EE &= S \sqcup a\mathsf{L}\neg S \sqcup a\mathsf{L}\neg S\mathsf{L}a \sqcup a\mathsf{L}\neg S\mathsf{L}\neg S \sqcup \neg S\mathsf{L}a \sqcup \neg S\mathsf{L}a\mathsf{L}\neg S \sqcup \\ &\quad\ \neg S\mathsf{L}\neg S\mathsf{L}a \sqcup \neg S\mathsf{L}\neg S\mathsf{L}\neg S \\ &\sqsubseteq S \sqcup a\mathsf{L}\neg S \sqcup a\mathsf{L}a \sqcup \neg S\mathsf{L}a \sqcup \neg S\mathsf{L}\neg S = E \sqcup a\mathsf{L}a = E \sqcup a = E \end{aligned}$$

Hence $p : A \leftrightarrow A/E$ exists with $pp^\mathsf{T} = E$ and $p^\mathsf{T} p = \mathsf{I}$. We show that A/E is a subset of A corresponding to S. To this end, define $i : A/E \leftrightarrow A$ by $i = p^\mathsf{T} S$. Then

- $ii^\mathsf{T} = p^\mathsf{T} SS^\mathsf{T} p = p^\mathsf{T} Sp \sqsubseteq p^\mathsf{T} p = \mathsf{I}$. Conversely, $p^\mathsf{T} p \sqsubseteq p^\mathsf{T} pp^\mathsf{T} Spp^\mathsf{T} p = p^\mathsf{T} Sp$ since $\mathsf{I} \sqsubseteq E = pp^\mathsf{T} Spp^\mathsf{T}$ using the Tarski rule:

$$pp^\mathsf{T} Spp^\mathsf{T} = ESE = (S \sqcup \neg S\mathsf{L}a)E = S \sqcup a\mathsf{L}\neg S \sqcup \neg S\mathsf{L}a \sqcup \neg S\mathsf{L}a\mathsf{L}\neg S = E$$

- $i^\mathsf{T} i = S^\mathsf{T} pp^\mathsf{T} S = SES = S(S \sqcup \neg S\mathsf{L}a) = S$. □

Proof (of backward implication of Theorem 20). Let $E : A \leftrightarrow A$ be an equivalence. Consider the relation $\sim$ on $\mathrm{at}_1(A)$ defined by $a \sim b \Leftrightarrow a\mathsf{L}b \sqsubseteq E$. It is an equivalence relation:

- $a \sim a$ since $a\mathsf{L}a = a \sqsubseteq \mathsf{I}$ using that a is rectangular.
- $a \sim b$ implies $b \sim a$ since $a\mathsf{L}b \sqsubseteq E$ implies $b\mathsf{L}a = (a\mathsf{L}b)^\mathsf{T} \sqsubseteq E^\mathsf{T} = E$.
- $a \sim b$ and $b \sim c$ imply $a \sim c$ since $a\mathsf{L}c = a\mathsf{L}b\mathsf{L}c = a\mathsf{L}bb\mathsf{L}c \sqsubseteq EE = E$ using the Tarski rule.

Let I be the equivalence classes of $\mathrm{at}_1(A)/{\sim}$ and let a_i be a representative of class $i \in I$. Define $S : A \leftrightarrow A$ by $S = \bigsqcup_{i\in I} a_i$. Then S is a non-zero partial identity since $\sim$ has at least one class. Hence $i : S \leftrightarrow A$ exists with $i^\mathsf{T} i = S$ and $ii^\mathsf{T} = \mathsf{I}$. We show that S is A/E. To this end, define $p : A \leftrightarrow S$ by $p = Ei^\mathsf{T}$. Then

- $pp^\mathsf{T} = Ei^\mathsf{T} iE^\mathsf{T} = ESE \sqsubseteq EE = E$. Conversely, $E = EE \sqsubseteq EESEE = ESE$ since $\mathsf{I} \sqsubseteq ESE$. To obtain the latter we show $a \sqsubseteq ESE$ for each $a \in \mathrm{at}_1(A)$. Since $ESE = E(\bigsqcup_{i\in I} a_i)E = \bigsqcup_{i\in I} Ea_iE$ it suffices to show $a \sqsubseteq Ea_iE$ using the representative a_i with $a \sim a_i$. This holds since $a \sqsubseteq a\mathsf{L}a = a\mathsf{L}a_i\mathsf{L}a = a\mathsf{L}a_ia_ia_i\mathsf{L}a \sqsubseteq Ea_iE$ using the Tarski rule.
- $\mathsf{I} = ii^\mathsf{T} \sqsubseteq iEi^\mathsf{T} = iE^\mathsf{T} Ei^\mathsf{T} = p^\mathsf{T} p$. Conversely, we have $iEi^\mathsf{T} = ii^\mathsf{T} iEi^\mathsf{T} ii^\mathsf{T} = iSESi^\mathsf{T} \sqsubseteq ii^\mathsf{T}$ since

$$SES = (\bigsqcup_{i\in I} a_i)E(\bigsqcup_{j\in I} a_j) = \bigsqcup_{i,j\in I} a_iEa_j \sqsubseteq \mathsf{I}$$

  for which it remains to show $a_iEa_j \sqsubseteq \mathsf{I}$. If $i = j$, then $a_iEa_i \sqsubseteq a_i\mathsf{L}a_i = a_i \sqsubseteq \mathsf{I}$ since a_i is rectangular. If $i \neq j$ we show $a_iEa_j = \mathsf{O}$. This is equivalent to $a_i\mathsf{L}a_j \sqsubseteq \overline{E}$ using Lemma 2.2. Since $a_i\mathsf{L}a_j$ is an atom by Lemma 19, the latter is equivalent to $a_i\mathsf{L}a_j \not\sqsubseteq E$, that is, to $a_i \not\sim a_j$, which holds since a_i and a_j represent different equivalence classes in this case. □

7 Independence of Domain Constructions

In this section we give models which show that there are no dependences between the studied domain constructions apart from those proved in the previous sections, under the assumption that all atoms are rectangular. All of the following models are (heterogeneous) subalgebras of REL, where the objects are down-closed subsets of the natural numbers $\mathbb{N}$, and the morphisms are all relations between these sets. For $n \in \mathbb{N}$ let $\mathbf{n}$ denote the set $\{0, 1, 2, \ldots, n-1\}$ of numbers smaller than n; for example $\mathbf{1} = \{0\}$, $\mathbf{2} = \{0, 1\}$ and $\mathbf{5} = \{0, 1, 2, 3, 4\}$.

The following table gives the models or why there are none, for each combination of having all power sets, products, sums and subsets. By Theorem 20 we do not distinguish between subsets and quotients. Due to lack of space we omit proofs that the models have or do not have the indicated domain constructions.

power	product	sum	subset	objects
no	no	no	no	$\mathbf{2}$
no	no	no	yes	$\mathbf{1}, \mathbf{2}$
no	no	yes	no	no model by Theorem 18
no	no	yes	yes	no model by Theorem 18
no	yes	no	no	$\mathbf{1}, \mathbb{N}$
no	yes	no	yes	$\mathbf{1}$
no	yes	yes	no	$\mathbb{N}$
no	yes	yes	yes	$\mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots, \mathbb{N}$
yes	no	no	no	$\mathbf{2^i}, \mathbf{3^i}$ for $i \in \mathbb{N}$
yes	no	no	yes	no model by Corollary 13 or Corollary 17
yes	no	yes	no	no model by Theorem 18
yes	no	yes	yes	no model by Corollary 17 or Theorem 18
yes	yes	no	no	$\mathbf{2^i}$ for $i \in \mathbb{N}$
yes	yes	no	yes	no model by Corollary 13
yes	yes	yes	no	$\mathbf{2}, \mathbf{3}, \mathbf{4}, \ldots$
yes	yes	yes	yes	$\mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots$

8 Conclusion

We have shown a number of dependences between the domain constructions of power sets, products, sums, quotients and subsets in heterogeneous relation algebras. Some results assumed that objects are comparable or that atoms are rectangular. This raises questions for further study:

- Can sums be constructed without assuming comparability?
- Can products be constructed without assuming rectangular atoms?
- How are subsets and quotients related without assuming rectangular atoms?

The second question refers to products as axiomatised in this paper, which implies representability. A weaker version of relational products that does not imply representability was investigated in [18, 19]. These papers also relate the relational product to the categorical product in the subcategory of mappings.

A referee noted that [17] shows ‘every (small) heterogeneous relation algebra can be faithfully embedded into a relation algebra that has relational sums and powers so that these constructions can always be generated’. The present paper does not embed into another algebra but shows the existence of domain constructions within a heterogeneous relation algebra under certain assumptions.

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References

1. Adámek, J., Herrlich, H., Strecker, G.E.: Abstract and Concrete Categories: The Joy of Cats. Dover Publications, second edn. (2009)
2. Berghammer, R., Schmidt, G., Zierer, H.: Symmetric quotients and domain constructions. Inf. Process. Lett. **33**(3), 163–168 (1989)
3. Berghammer, R., Winter, M.: Embedding mappings and splittings with applications. Acta Inf. **47**(2), 77–110 (2010)
4. Bird, R., de Moor, O.: Algebra of Programming. Prentice Hall (1997)
5. Desharnais, J.: Abstract Relational Semantics. Ph.D. thesis, McGill University (1989)
6. Freyd, P.J., Ščedrov, A.: Categories, Allegories, North-Holland Mathematical Library, vol. 39. Elsevier Science Publishers (1990)
7. Givant, S., Halmos, P.: Introduction to Boolean Algebras. Springer (2009)
8. Guttmann, W.: Multirelations with infinite computations. Journal of Logical and Algebraic Methods in Programming **83**(2), 194–211 (2014)
9. Maddux, R.D.: Pair-dense relation algebras. Transactions of the American Mathematical Society **328**(1), 83–131 (1991)
10. Olivier, J.P., Serrato, D.: Catégories de Dedekind: Morphismes transitifs dans les catégories de Schröder. Comptes rendus hebtomadaires des séances de l’Académie des Sciences, Série A **290**, 939–941 (1980)
11. Schmidt, G.: Relational Mathematics. Cambridge University Press (2011)
12. Schmidt, G., Hattensperger, C., Winter, M.: Heterogeneous relation algebra. In: Brink, C., Kahl, W., Schmidt, G. (eds.) Relational Methods in Computer Science, chap. 3, pp. 39–53. Springer, Wien (1997)
13. Schmidt, G., Ströhlein, T.: Relationen und Graphen. Springer (1989)
14. Schmidt, G., Winter, M.: Relational Topology, Lecture Notes in Mathematics, vol. 2208. Springer (2018)
15. Tarski, A.: On the calculus of relations. The Journal of Symbolic Logic **6**(3), 73–89 (1941)
16. Tarski, A., Givant, S.: A formalization of set theory without variables, Colloquium Publications, vol. 41. American Mathematical Society (1987)
17. Winter, M.: Strukturtheorie heterogener Relationenalgebren mit Anwendung auf Nichtdeterminismus in Programmiersprachen. Ph.D. thesis, Universität der Bundeswehr München (1998)
18. Winter, M.: Weak relational products. In: Schmidt, R.A. (ed.) RelMiCS/AKA 2006. LNCS, vol. 4136, pp. 417–431. Springer (2006)
19. Winter, M.: Products in categories of relations. Journal of Logic and Algebraic Programming **76**(1), 145–159 (2008)
20. Zierer, H.: Relation algebraic domain constructions. Theor. Comput. Sci. **87**, 163–188 (1991)