An Algebraic Approach to Multirelations and their Properties

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Abstract

We study operations and equational properties of multirelations, which have been used for modelling games, protocols, computations, contact, closure and topology. The operations and properties are expressed using sets, heterogeneous relation algebras and more general algebras for multirelations. We investigate the algebraic properties of a new composition operation based on the correspondence to predicate transformers, different ways to express reflexive-transitive closures of multirelations, numerous equational properties, how these properties are connected and their preservation by multirelational operations. We particularly aim to generalise results and properties from up-closed multirelations to arbitrary multirelations. This paper is an extended version of [7].

Keywords: algebras of multirelations, Aumann contact, heterogeneous relations, multirelational composition, reflexive-transitive closure

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1. Introduction

A relation between two sets A and B is a subset of the Cartesian product $A \times B$. Reasoning about relations can be done using this set-based definition or, more abstractly, using (heterogeneous) relation algebras; for example, see [10, 34, 36]. One of the advantages of the algebraic approach is that many frequently used properties of relations can be expressed by concise equations or inequalities. For example, the relation R is transitive if and only if $RR \subseteq R$, using the composition of R with itself on the left-hand side of the inequality. In contrast, the usual set-based definition of transitivity involves three universally quantified variables. Because any inequality $Q \subseteq R$ can be translated to an equation $Q \cup R = R$ using the union of sets, we call such properties 'equational'. Similarly concise formulas can express properties of functions, orders, graphs and programs, which are often modelled by relations. Many examples can be found in [32, 35].

In this paper we work towards a compendium of properties for multirelations. A multirelation is a relation between a set A and the powerset 2^B of a set B. The additional powerset structure is used, for example, for modelling two-player games, the interaction between agents in a computation, and the topological notion of a contact; for example, see [1, 5, 23, 25]. Properties of multirelations appear in the literature typically in a set-based form. More recently, researchers have started to consider multirelations from an algebraic perspective, for example, in [18, 19, 21, 22]. It is therefore a natural step to try to express multirelational properties algebraically, to find out how they are connected and which algebras are suitable for reasoning about them.

Studying properties of multirelations is not a straight-forward generalisation of existing work on relations. Multirelations differ from arbitrary relations by using the powerset structure on their targets. As a consequence, they support operations – such as the multirelational dual – which are not available for

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arbitrary relations, and fail to support operations – such as converse – which are available for general relations. Unlike relational composition, which is a standard notion, at least four ways to sequentially compose multirelations have been studied in the literature; we will look at two of these in the present paper. Some of these composition operations fail to satisfy basic distributivity and associativity properties taken for granted for relational composition. While the algebra of relations has been studied at least since A. Tarski's axiomatisation 75 years ago in [36], the algebra of multirelations is much less well understood beyond its basic properties.

The properties considered in the present paper comprise several from the existing literature, others which are formally similar to properties of relations, properties obtained by formal dualisation, and further properties that turned out to be useful during our investigation. One of these properties describes up-closed multirelations, which are required in many previous works; a key achievement of the present paper is to show that many results about multirelations can be derived without a restriction to up-closed multirelations. Our method of study is mostly algebraic, in particular, as regards the relationships between properties as well as their preservation by multirelational operations. Many results in this paper have been verified using Isabelle/HOL and its integrated automated theorem provers and SMT solvers. Numerous counterexamples are provided to show that certain operations do not preserve certain properties; for the most part, these have been generated by a Haskell program. Besides the equational properties of multirelations, we study the algebraic properties of a new composition operation, relate various ways of describing reflexive-transitive closures of multirelations, and present a connection of multirelations to contact relations introduced by G. Aumann in [1].

An overview of this paper follows. In Sections 2 and 3 we start by representing multirelations and their operations in terms of relation algebras. We recall fundamental algebraic properties of multirelations and, in particular, of the composition operation introduced by R. Parikh in [25]. In Section 4 we show many algebraic properties of a different composition operation, which we have recently introduced based on a correspondence to predicate transformers in [8]. To further abstract from the relation-algebraic representation we introduce more general algebras in Section 5. They are based on Boolean algebras and semirings; their axioms capture fundamental properties of multirelational operations. In Section 6 we relate different definitions of reflexive-transitive closures in a very general algebraic setting which covers arbitrary multirelations. The above-mentioned equational properties of multirelations are studied in Section 7; in particular, we show numerous relationships between these properties. Section 8 is concerned with the preservation of these properties by multirelational operations. Our results here are complete in the sense that for each property and each fundamental operation we either prove that the property is preserved or falsify this by providing a counterexample. In Section 9 we discuss the connection to Aumann contact relations. Finally, Section 10 gives new logical characterisations of two distributivity properties of multirelations.

Overall, this paper introduces algebraic structures which capture arbitrary multirelations and uses these structures to study reflexive-transitive closure operations as well as equational properties of multirelations, their relationships and their preservation by multirelational operations. In addition, the paper gives logical representations of the properties and studies a composition operation recently introduced in [8]. In that companion paper we have investigated how some of the properties discussed here translate to predicate transformers. This was facilitated by a relation-algebraic correspondence between multirelations and predicate transformers, which is similar to the correspondence between contact relations and closure operations.

The contributions of this extended version with respect to the first version [7] are

1. a relation-algebraic investigation of the properties of an alternative composition operation of multirelations defined in [8] (Section 4);
2. additional properties of zero-vectors, one-vectors and down-closed multirelations, an investigation of their relationships with other properties and their preservation by multirelational operations, and the verification of the results in Isabelle/HOL (Sections 7.1 and 7.2 and Figures 1, 2, 3 and 6);
3. an extension of algebraic structures for multirelations by a complement operation and an investigation of which properties it preserves, again verified in Isabelle/HOL (Section 7.2 and Figure 7);
4. answers to the open questions in [7] regarding the preservation of $\cup$- and $\cap$-distributivity, and thereby a complete decision of which properties are preserved by which operations (Figure 5 and Theorem 21);

5. characterisations of arbitrary $\cup$- and $\cap$-distributive multirelations including a new, weak finiteness condition, and derivation of two previous results as special cases (Section 10).

Most parts of the remaining Sections 2, 3, 5, 6, 8 and 9 are taken from the first version [7] with only small changes to reflect our new results.

2. Relation-Algebraic Prerequisites

In this section we present the facts about relations and heterogeneous relation algebras that are needed in the remainder of this paper. For more details on relations and relation algebras, see [32], for example.

We write $R : A \leftrightarrow B$ if R is a (typed binary) relation with source A and target B, that is, of type $A \leftrightarrow B$. If the sets A and B are finite, we may consider R as a Boolean matrix. Since this interpretation is well suited for many purposes, we will use matrix notation and write $R_{x,y}$ instead of $(x, y) \in R$ or $x\,R\,y$.

We assume the reader to be familiar with the basic operations on relations, namely R^{c} (converse), $\overline{R}$ (complement), $R \cup S$ (union), $R \cap S$ (intersection), RS (composition), the predicates $R \subseteq S$ (inclusion) and $R = S$ (equality) and the special relations O (empty relation), T (universal relation) and I (identity relation). Converse has higher precedence than composition, which has higher precedence than union and intersection. The set of all relations of type $A \leftrightarrow B$ with the operations $\overline{}$, $\cup$, $\cap$, the ordering $\subseteq$ and the constants O and T forms a complete Boolean lattice. Further well-known rules are, for example, $(R^{\mathsf{c}})^{\mathsf{c}} = R$, $(QR)^{\mathsf{c}} = R^{\mathsf{c}}Q^{\mathsf{c}}$, $\overline{R^{\mathsf{c}}} = \overline{R}^{\mathsf{c}}$, and that $R \subseteq S$ implies $R^{\mathsf{c}} \subseteq S^{\mathsf{c}}$ as well as $RP \subseteq SP$ and $QR \subseteq QS$, for all P, Q, R and S with appropriate types. In the remainder of this paper we assume that all relational expressions and formulas are correctly typed.

The theoretical framework for these rules and many others is that of a (heterogeneous) relation algebra; see [34] for details. As constants and operations of this algebraic structure we have those of concrete (that is, set-theoretic) relations. The axioms of a relation algebra are those of a complete Boolean lattice for the Boolean part, the associativity and neutrality of identity relations for composition, the equivalence of $QR \subseteq S$ and $Q^{\mathsf{c}}\overline{S} \subseteq \overline{R}$ and $\overline{S}R^{\mathsf{c}} \subseteq \overline{Q}$, for all relations Q, R and S (called the *Schröder equivalences*) and that $R \neq \mathsf{O}$ implies $\mathsf{T}R\mathsf{T} = \mathsf{T}$, for all relations R (called the *Tarski rule*).

A relation R is *injective* if $RR^{\mathsf{c}} \subseteq \mathsf{I}$, *surjective* if $\mathsf{T}R = \mathsf{T}$ and *bijective* if R is injective and surjective. If R is injective, then $(P \cap Q)R = PR \cap QR$, for all relations P and Q. If R is bijective, then $\overline{QR} = \overline{Q}R$ and $PR^{\mathsf{c}} \subseteq Q$ is equivalent to $P \subseteq QR$, for all relations P and Q. In general, the relation R^{c} cannot be brought into a composition on the other side of $PR^{\mathsf{c}} \subseteq Q$, but this works by using residuals.

Residuals are the greatest solutions of certain relation-algebraic inclusions. The *left residual* of S over R, in symbols S/R, is the greatest relation X such that $XR \subseteq S$. So, we have the Galois connection $XR \subseteq S$ if and only if $X \subseteq S/R$, for all relations X. Similarly, the *right residual* of S over R, in symbols $R\backslash S$, is the greatest relation X such that $RX \subseteq S$. This implies that $RX \subseteq S$ if and only if $X \subseteq R\backslash S$, for all relations X. We will also need relations which are left and right residuals simultaneously. The *symmetric quotient* $R \div S$ of two relations R and S is defined as the greatest relation X such that $RX \subseteq S$ and $XS^{\mathsf{c}} \subseteq R^{\mathsf{c}}$. In terms of the basic operations we have $S/R = \overline{\overline{S}R^{\mathsf{c}}}$ and $R\backslash S = \overline{R^{\mathsf{c}}\overline{S}}$ and $R \div S = (R\backslash S) \cap (R^{\mathsf{c}}/S^{\mathsf{c}})$, for all relations R and S.

We use the following basic properties of residuals and symmetric quotients:

(1) $Q(Q\backslash R) \subseteq R$.	(5) $(Q\backslash R)P = Q\backslash(RP)$ if P is bijective.
(2) $Q \div \mathsf{O} = Q\backslash \mathsf{O}$.	(6) $(Q/R)P = Q/(P^{\mathsf{c}}R)$ if P is bijective.
(3) $(Q \div R)^{\mathsf{c}} = R \div Q$.	(7) $(Q \div R)P = Q \div (RP)$ if P is bijective.
(4) $\overline{Q} \div \overline{R} = Q \div R$.	(8) $P^{\mathsf{c}}(Q \div R) = (QP) \div R$ if P is bijective.

See [9, 15, 35] for some of these and related properties. For example, a proof of (7) is

$$(Q \div R)P = ((Q\backslash R) \cap (Q^{\mathsf{c}}/R^{\mathsf{c}}))P = (Q\backslash R)P \cap (Q^{\mathsf{c}}/R^{\mathsf{c}})P = (Q\backslash(RP)) \cap (Q^{\mathsf{c}}/(P^{\mathsf{c}}R^{\mathsf{c}})) = Q \div (RP)$$

using (5) and (6). From (3) and (7) we obtain (8) by the calculation

$$P^{\mathsf{c}}(Q \div R) = ((Q \div R)^{\mathsf{c}}P)^{\mathsf{c}} = ((R \div Q)P)^{\mathsf{c}} = (R \div (QP))^{\mathsf{c}} = (QP) \div R.$$

Furthermore, we have as component-wise specification of symmetric quotients that $(Q \div R)_{x,y}$ if and only if $\forall z : Q_{z,x} \Leftrightarrow R_{z,y}$, for all x, y.

Besides empty relations, universal relations and identity relations, we need further basic relations which specify fundamental set-theoretic constructions. Assume that A is a set and let 2^A denote its powerset. Then the *membership relation* $\mathsf{E} : A \leftrightarrow 2^A$ is the relation-level equivalent of the set-theoretic predicate '$\in$'. Hence, we have $\mathsf{E}_{x,Y}$ if and only if $x \in Y$, for all $x \in A$ and $Y \in 2^A$. With the help of the relation E we can introduce two relations on 2^A via the definitions $\mathsf{S} := \mathsf{E}\backslash\mathsf{E} : 2^A \leftrightarrow 2^A$ and $\mathsf{C} := \mathsf{E} \div \overline{\mathsf{E}} : 2^A \leftrightarrow 2^A$. A little component-wise calculation shows $\mathsf{S}_{X,Y}$ if and only if $X \subseteq Y$ and $\mathsf{C}_{X,Y}$ if and only if $Y = \overline{X}$, for all $X, Y \in 2^A$, where $\overline{X}$ is the complement of the set X relative to its superset A. Therefore, we call S a *subset relation* and C a *set complement relation*. We use the following basic properties of the membership, subset and complement relations:

(1)	$\mathsf{E} \div \mathsf{E} = \mathsf{I}$.	(6)	$\mathsf{SS} \subseteq \mathsf{S}$.
(2)	$\mathsf{E} \div R$ is bijective.	(7)	$\mathsf{S}^{\mathsf{c}}\mathsf{C} = \mathsf{CS}$.
(3)	$(Q \div \mathsf{E})(\mathsf{E} \div R) = Q \div R$.	(8)	$\mathsf{C}^{\mathsf{c}} = \mathsf{C}$.
(4)	$\mathsf{S}(\mathsf{E} \div R) = \mathsf{E}\backslash R$.	(9)	C is bijective.
(5)	$\mathsf{I} \subseteq \mathsf{S}$.	(10)	$\overline{\mathsf{E}} = \mathsf{EC}$.

See [15, 21, 31] for these and related properties of the relations E, S and C.

3. Fundamentals of Multirelations

In this section we recall basic definitions, operations and properties of multirelations and express them in terms of relations. The presentation follows [21].

A *multirelation* (as introduced in [25, 29]) is a relation of type $A \leftrightarrow 2^B$ in the sense of Section 2. It maps an element of A to a set of subsets of B. Union, intersection and complement apply to multirelations as to relations. Particular multirelations are the empty relations $\mathsf{O} : A \leftrightarrow 2^B$, the universal relations $\mathsf{T} : A \leftrightarrow 2^B$ and the membership relations $\mathsf{E} : A \leftrightarrow 2^A$. The composition of the multirelations $Q : A \leftrightarrow 2^B$ and $R : B \leftrightarrow 2^C$ proposed by Parikh is the multirelation $Q \,;\! R : A \leftrightarrow 2^C$ given by

$$(Q\,;\!R)_{x,Z} \iff (\exists Y \in 2^B : Q_{x,Y} \wedge \forall y \in Y : R_{y,Z}),$$

for all $x \in A$ and $Z \in 2^C$. The *dual* of a multirelation $R : A \leftrightarrow 2^B$ is the multirelation $R^{\mathsf{d}} : A \leftrightarrow 2^B$ given by

$$R^{\mathsf{d}}{}_{x,Y} \iff \neg R_{x,\overline{Y}},$$

for all $x \in A$ and $Y \in 2^B$, where $\overline{Y}$ is the complement of Y relative to its superset B. Dual has higher precedence than composition, which has higher precedence than union and intersection. A multirelation $R : A \leftrightarrow 2^B$ is *up-closed* if

$$R_{x,Y} \wedge Y \subseteq Z \implies R_{x,Z}$$

for all $x \in A$ and $Y, Z \in 2^B$. This means that if an element of A is related to a set Y, it also has to be related to all supersets of Y. By $A \overset{\mathsf{u}}{\leftrightarrow} 2^B$ we denote the set of all up-closed multirelations of type $A \leftrightarrow 2^B$.

The following result expresses multirelational composition, the dual and the property of being up-closed in terms of relation-algebraic operations and constants, namely right residual, membership relation E, set complement relation C and subset relation S. It is proved in [21, Theorems 2, 4 and 6]; see also [22, 32].

Theorem 1. *Let $Q : A \leftrightarrow 2^B$ and $R : B \leftrightarrow 2^C$ be multirelations. Then we have $Q \,;\! R = Q(\mathsf{E}\backslash R)$ and $Q^{\mathsf{d}} = \overline{Q}\mathsf{C} = \overline{Q\mathsf{C}}$. Furthermore, Q is up-closed if and only if $Q = Q\mathsf{S}$.*

A multirelation $R : A \leftrightarrow 2^A$ models a two-player game as shown in [25]. The set A describes the possible states of the game. For each state $x \in A$ the set of subsets $Ys = \{Y \in 2^A \mid R_{x,Y}\}$ to which x is related gives the options of the first player. The first player chooses one of these subsets, a set $Y \in Ys$. This set Y gives the options of the second player, who chooses one of its elements $y \in Y$, which is the next state of the game. If the first player cannot make a choice because Ys is empty, the second player wins. If the second player cannot make a choice because Y is empty, the first player wins. Multirelations can also be used to describe the interaction of two agents in a computation (see [5, 13, 23]), certain kinds of contact (see [1, 4]) and concurrency (see [27]).

Being relations, the multirelations of type $A \leftrightarrow 2^B$ form a bounded distributive lattice under the operations of union and intersection. The structure becomes more diversified once we take composition into account. First, familiar laws of relation algebras – that composition distributes over union and has the empty relation as a zero – no longer hold from both sides, but just from one side. Second, other laws of relation algebras – that composition is associative and has the identity relation as a neutral element – hold for up-closed multirelations, but need to be weakened in the general case as shown in [17]. On the other hand, composition remains $\subseteq$-isotone. These and related properties are summarised in the following result.

Theorem 2. *For all multirelations P, Q and R we have*

$$\textit{(1)}\ \mathsf{O}\,;R = \mathsf{O} \qquad \textit{(2)}\ \mathsf{E}\,;R = R \qquad \textit{(3)}\ \mathsf{T}\,;R = \mathsf{T} \qquad \textit{(4)}\ R \subseteq R\,;\mathsf{E},$$

where in (4) equality holds if and only if R is up-closed, and also

$$\textit{(5)}\ (P \cup Q)\,;R = P\,;R \cup Q\,;R \qquad \textit{(6)}\ (P \cap Q)\,;R \subseteq P\,;R \cap Q\,;R,$$

where in (6) equality holds if P and Q are up-closed, and also

$$\textit{(7)}\ (P\,;Q)\,;R \subseteq P\,;(Q\,;R),$$

where in (7) equality holds if Q is up-closed, and finally

$$\textit{(8)}\ P\,;Q \cup P\,;R \subseteq P\,;(Q \cup R) \qquad \textit{(9)}\ P\,;(Q \cap R) \subseteq P\,;Q \cap P\,;R.$$

Proof. All properties are proved in [21, Theorems 3 and 7] except (4) and (7) for general multirelations. A proof of (4) is $R \subseteq R\mathsf{S} = R(\mathsf{E}\backslash\mathsf{E}) = R\,;\mathsf{E}$. To prove (7) we use that $\mathsf{E}(\mathsf{E}\backslash Q)(\mathsf{E}\backslash R) \subseteq Q(\mathsf{E}\backslash R)$ implies $(\mathsf{E}\backslash Q)(\mathsf{E}\backslash R) \subseteq \mathsf{E}\backslash(Q(\mathsf{E}\backslash R))$ by the Galois connection of right residuals. Hence, by

$$(P\,;Q)\,;R = (P\,;Q)(\mathsf{E}\backslash R) = P(\mathsf{E}\backslash Q)(\mathsf{E}\backslash R) \subseteq P(\mathsf{E}\backslash(Q(\mathsf{E}\backslash R))) = P(\mathsf{E}\backslash(Q\,;R)) = P\,;(Q\,;R)$$

we get the desired result. □

The dual operation reverses the lattice order and distributes over composition of up-closed multirelations. Again this needs to be weakened in the general case. These and further properties are summarised in the following result.

Theorem 3. *For all multirelations Q and R we have*

$$\textit{(1)}\ \mathsf{O}^{\mathsf{d}} = \mathsf{T} \qquad \textit{(2)}\ \mathsf{E}^{\mathsf{d}} = \mathsf{E} \qquad \textit{(3)}\ \mathsf{T}^{\mathsf{d}} = \mathsf{O} \qquad \textit{(4)}\ R^{\mathsf{dd}} = R,$$

and also

$$\begin{array}{ll} \textit{(5)}\ (Q \cup R)^{\mathsf{d}} = Q^{\mathsf{d}} \cap R^{\mathsf{d}} & \textit{(7)}\ (Q\,;R)^{\mathsf{d}} \subseteq Q^{\mathsf{d}}\,;R^{\mathsf{d}} \\ \textit{(6)}\ (Q \cap R)^{\mathsf{d}} = Q^{\mathsf{d}} \cup R^{\mathsf{d}} & \textit{(8)}\ (Q\,;R)^{\mathsf{d}} = (Q\,;\mathsf{E})^{\mathsf{d}}\,;R^{\mathsf{d}}, \end{array}$$

where in (7) equality holds if Q is up-closed.

Proof. All properties are proved in [21, Theorems 5 and 7] except (7) and (8) for general multirelations. For proving (7) and (8) we use properties of the relations E, S and C and of symmetric quotients given in Section 2 and reason as follows:

$$\begin{aligned} (Q\,;R)^{\mathsf{d}} &= \overline{Q\,;R}\mathsf{C} = \overline{Q(\mathsf{E}\backslash R)}\mathsf{C} = \overline{Q\mathsf{S}(\mathsf{E}\div R)}\mathsf{C} = \overline{Q\mathsf{S}}(\mathsf{E}\div R)\mathsf{C} = \overline{Q\mathsf{S}}(\overline{\mathsf{E}}\div\overline{R})\mathsf{C} \\ &= \overline{Q\mathsf{S}}(\overline{\mathsf{E}}\div\mathsf{E})(\mathsf{E}\div\overline{R})\mathsf{C} = \overline{Q\mathsf{S}}((\mathsf{E}\mathsf{C})\div\mathsf{E})(\mathsf{E}\div\overline{R})\mathsf{C} = \overline{Q\mathsf{S}}\mathsf{C}^{\mathsf{c}}(\mathsf{E}\div\mathsf{E})(\mathsf{E}\div\overline{R})\mathsf{C} \\ &= \overline{Q\mathsf{S}}\mathsf{C}(\mathsf{E}\div\overline{R})\mathsf{C} = \overline{Q\mathsf{S}}\mathsf{S}^{\mathsf{c}}\mathsf{C}(\mathsf{E}\div\overline{R})\mathsf{C} = \overline{Q\mathsf{S}}\mathsf{C}\mathsf{S}(\mathsf{E}\div\overline{R})\mathsf{C} = \overline{Q\mathsf{S}}\mathsf{C}(\mathsf{E}\backslash\overline{R})\mathsf{C} \\ &= (Q\mathsf{S})^{\mathsf{d}}(\mathsf{E}\backslash\overline{R})\mathsf{C} = (Q\mathsf{S})^{\mathsf{d}}(\mathsf{E}\backslash(\overline{R}\mathsf{C})) = (Q(\mathsf{E}\backslash\mathsf{E}))^{\mathsf{d}}(\mathsf{E}\backslash R^{\mathsf{d}}) = (Q\,;\mathsf{E})^{\mathsf{d}}\,;R^{\mathsf{d}} \subseteq Q^{\mathsf{d}}\,;R^{\mathsf{d}} \end{aligned}$$

This calculation also uses $\overline{Q\mathsf{S}}\mathsf{S}^{\mathsf{c}} = \overline{Q\mathsf{S}}$. The inclusion '$\subseteq$' of this equality follows by applying a Schröder equivalence to $Q\mathsf{S}\mathsf{S} \subseteq Q\mathsf{S}$ and the inclusion '$\supseteq$' follows from $\mathsf{I} \subseteq \mathsf{S}$. □

4. Alternative Compositions of Multirelations

Sequential composition has a standard definition for relations, but for multirelations the situation is less satisfying. Several definitions of a multirelational composition have been proposed. Because none of them is canonical it would be useful to find at least some common algebraic properties. This motivates the present section that discusses different ways in which multirelations can be composed sequentially. We have seen one of them in the previous section.

The operation $Q\,;R = Q(\mathsf{E}\backslash R)$ suggested by Parikh in [25] is not the only way to define a composition of multirelations. In [27] D. Peleg uses multirelations to model aspects of concurrency and defines a suitable composition; see also [18]. In [16] these compositions and a composition based on Kleisli categories are investigated using liftings of multirelations.

Parikh's and Peleg's composition operations are not associative for arbitrary multirelations, and therefore cannot be used to define a category of all multirelations. However, predicate transformers form a category and there is a one-to-one correspondence between predicate transformers and multirelations. Specifically, for a multirelation $R : A \leftrightarrow 2^B$, the relation $\Psi(R) = (R{\div}\mathsf{E}) : 2^B \to 2^A$ is a predicate transformer. Conversely, from a predicate transformer $P : 2^B \to 2^A$, we obtain the multirelation $\Phi(P) = \mathsf{E}P^{\mathsf{c}} : A \leftrightarrow 2^B$. In these constructions a predicate transformer, which is a mapping of type $2^B \to 2^A$, is identified with the corresponding relation of type $2^B \leftrightarrow 2^A$. Moreover, the functions Ψ and Φ are mutually inverse.

In [8] we have therefore introduced an alternative composition of multirelations using this correspondence. For multirelations $Q : A \leftrightarrow 2^B$ and $R : B \leftrightarrow 2^C$ we define the composition $Q\,{:}\,R$ by

$$(Q\,{:}\,R) = \Phi(\Psi(R)\Psi(Q)) = Q(\mathsf{E}{\div}R).$$

This definition of composition is similar to Parikh's composition, but uses the symmetric quotient instead of the right residual. In predicate logic, it therefore amounts to

$$(Q\,{:}\,R)_{x,Z} \iff (\exists Y \in 2^B : Q_{x,Y} \wedge \forall y \in B : y \in Y \Leftrightarrow R_{y,Z}),$$

for all $x \in A$ and $Z \in 2^C$. Parikh's composition requires only $y \in Y \Rightarrow R_{y,Z}$, not the backward implication. Letting $R^{\mathsf{c}}(Z) = \{y \in B \mid R_{y,Z}\}$ be the set of elements that R relates to Z, we obtain $(Q\,{:}\,R)_{x,Z}$ if and only if $Q_{x,R^{\mathsf{c}}(Z)}$. In the following we investigate fundamental algebraic properties of this alternative composition operation.

Theorem 4. *For all multirelations P, Q and R we have:*

(1) $Q\,{:}\,\mathsf{E} = Q$.
(2) $Q\,;R = Q\mathsf{S}\,{:}\,R$.
(3) Q *is up-closed if and only if* $Q\,{:}\,R = Q\,;R$ *for all* R.
(4) $\mathsf{O}\,{:}\,R = \mathsf{O}$.
(5) $\mathsf{E}\,{:}\,R = R$.
(6) $\mathsf{T}\,{:}\,R = \mathsf{T}$.
(7) $Q\,{:}\,\mathsf{O} = Q\,;\mathsf{O}$.
(8) $(P \cup Q)\,{:}\,R = P\,{:}\,R \cup Q\,{:}\,R$.
(9) $\lambda X.(X\,{:}\,R)$ *is* $\subseteq$*-isotone.*
(10) Q *is up-closed if and only if* $\lambda X.(Q\,{:}\,X)$ *is* $\subseteq$*-isotone.*
(11) $(P\,{:}\,Q)\,{:}\,R = P\,{:}\,(Q\,{:}\,R)$.
(12) $\overline{Q\,{:}\,R} = \overline{Q}\,{:}\,R$.
(13) $(Q\,{:}\,R)P = Q\,{:}\,(RP)$ *if* P *is bijective.*
(14) $(Q\,{:}\,R)^{\mathsf{d}} = Q^{\mathsf{d}}\,{:}\,R^{\mathsf{d}}$.

Proof. We use properties of E, S, C and symmetric quotients given in Section 2. Property (1) holds by

$$Q\,{:}\,\mathsf{E} = Q(\mathsf{E}{\div}\mathsf{E}) = Q\mathsf{I} = Q$$

and property (2) follows from the calculation

$$Q\mathsf{S}\,{:}\,R = Q\mathsf{S}(\mathsf{E}{\div}R) = Q(\mathsf{E}\backslash R) = Q\,;R.$$

This entails the forward implication of property (3) since $Q = Q\mathsf{S}$ for up-closed Q by Theorem 1. For the backward implication of property (3) we instantiate the given equality with $R = \mathsf{E}$ to get $Q = Q\,{:}\,\mathsf{E} = Q\,;\mathsf{E}$ using (1). From this the claim follows by Theorem 2.

Next, properties (4), (5) and (6) follow by property (3) and Theorem 2 since O, E and T are up-closed multirelations. Property (7) is shown by

$$Q\,{:}\,\mathsf{O} = Q(\mathsf{E}{\div}\mathsf{O}) = Q(\mathsf{E}\backslash\mathsf{O}) = Q\,;\mathsf{O},$$

property (8) by

$$(P\cup Q)\,{:}\,R = (P\cup Q)(\mathsf{E}{\div}R) = P(\mathsf{E}{\div}R)\cup Q(\mathsf{E}{\div}R) = P\,{:}\,R\cup Q\,{:}\,R$$

and property (9) immediately follows from property (8) since distributive operations are $\subseteq$-isotone.

Next, we show property (10). The forward implication of this equivalence follows by property (3) and Theorem 2. For the backward implication, let $Q : A \leftrightarrow 2^B$ and assume that the function $\lambda X.(Q : X)$ is $\subseteq$-isotone. To show that Q is up-closed, let $x \in A$ and $Y, Z \in 2^B$ be given such that $Q_{x,Y}$ and $Y \subseteq Z$. We need to show $Q_{x,Z}$. To this end, we define multirelations $P, R : B \leftrightarrow 2^B$ as follows:

$$P := \{(x, X) \mid x \in Y \wedge X \in 2^B\} \qquad\qquad R := \{(x, X) \mid x \in Z \wedge X \in 2^B\}$$

Then we get $P \subseteq R$, whence $Q : P \subseteq Q : R$ by the assumption, and therefore $Q(\mathsf{E}{\div}P) \subseteq Q(\mathsf{E}{\div}R)$. Since $\mathsf{E}{\div}R$ is a bijective relation, we obtain

$$Q(\mathsf{E}{\div}P)(R{\div}\mathsf{E}) = Q(\mathsf{E}{\div}P)(\mathsf{E}{\div}R)^{\mathsf{c}} \subseteq Q.$$

To show $Q_{x,Z}$ it therefore suffices to show $(Q(\mathsf{E}{\div}P)(R{\div}\mathsf{E}))_{x,Z}$. Due to the assumption $Q_{x,Y}$, this follows if we can verify $(\mathsf{E}{\div}P)_{Y,\emptyset}$ and $(R{\div}\mathsf{E})_{\emptyset,Z}$. The first condition amounts to the equivalence of $\mathsf{E}_{y,Y}$ and $P_{y,\emptyset}$, for all y, and this is true by the definitions of E and P. The second condition amounts to the equivalence of $R_{y,\emptyset}$ and $\mathsf{E}_{y,Z}$, for all y, which holds by the definitions of R and E. This completes the proof of (10).

Since $\mathsf{E}{\div}R$ is a bijective relation, property (11) follows by

$$(P\,{:}\,Q)\,{:}\,R = P(\mathsf{E}{\div}Q)(\mathsf{E}{\div}R) = P(\mathsf{E}{\div}(Q(\mathsf{E}{\div}R))) = P\,{:}\,(Q\,{:}\,R)$$

and property (12) holds by

$$\overline{Q\,{:}\,R} = \overline{Q(\mathsf{E}{\div}R)} = \overline{Q}(\mathsf{E}{\div}R) = \overline{Q}\,{:}\,R.$$

For property (13), let P be bijective. Then the result follows by

$$(Q\,{:}\,R)P = Q(\mathsf{E}{\div}R)P = Q(\mathsf{E}{\div}(RP)) = Q\,{:}\,(RP).$$

Finally, property (14) is proved by the calculation

$$\begin{aligned}(Q\,{:}\,R)^{\mathsf{d}} &= \overline{Q\,{:}\,R}\mathsf{C} = (\overline{Q}\,{:}\,R)\mathsf{C} = \overline{Q}\,{:}\,(R\mathsf{C}) = \overline{Q}\,{:}\,\overline{R}^{\mathsf{d}} = \overline{Q}(\mathsf{E}{\div}\overline{R}^{\mathsf{d}}) = \overline{Q}(\overline{\mathsf{E}}{\div}R^{\mathsf{d}}) \\ &= \overline{Q}((\mathsf{E}\mathsf{C}){\div}R^{\mathsf{d}}) = \overline{Q}\mathsf{C}^{\mathsf{c}}(\mathsf{E}{\div}R^{\mathsf{d}}) = \overline{Q}\mathsf{C}(\mathsf{E}{\div}R^{\mathsf{d}}) = Q^{\mathsf{d}}(\mathsf{E}{\div}R^{\mathsf{d}}) = Q^{\mathsf{d}}\,{:}\,R^{\mathsf{d}},\end{aligned}$$

where we use properties (12) and (13). □

Compared to Parikh's composition, we thus obtain a monoid structure for the alternative composition as well as distributivity properties for the dual operation $^{\mathsf{d}}$ and the complement operation $^{-}$. On the other hand, the alternative composition operation is not $\subseteq$-isotone in its second argument. Both composition operations coincide for up-closed multirelations.

To further relate these operations, we compare when an element $x \in A$ is related to a set $Z \in 2^C$ by the compositions of $Q : A \leftrightarrow 2^B$ and $R : B \leftrightarrow 2^C$. Specifically we discuss the choices involved under the two-player interpretation given in Section 3. In this case, x is the starting state for player 1 and Z is the set of end states that player 1 tries to reach in two steps, one according to Q followed by one according to R. To this end, player 1 first chooses an intermediate set of states $Y \in 2^B$ that is related to x by Q. Then player 2 will choose one of the states $y \in Y$. Finally, the end states Z must be related to y by R so player 1 can choose this set. Depending on the composition and whether the multirelations are up-closed we obtain the following restrictions for the choice of Y:

- If Q and R are up-closed, consider the set Y' of all $y \in B$ such that $R_{y,W}$ for some $W \subseteq Z$. Then Y must be a subset of Y' to obtain $(Q\,;R)_{x,Z}$.
- If we cannot assume that Q and R are up-closed, we have to require $W = Z$; that is, let $Y' = R^{\mathsf{c}}(Z)$ be the set of all $y \in B$ such that $R_{y,Z}$. Still $Y \subseteq Y'$ is enough to obtain $(Q\,;R)_{x,Z}$ because given that Z can be reached from all $y \in Y'$, it can certainly be reached from all $y \in Y$.
- For the alternative composition, we use the same Y' as in the previous case because we cannot assume the multirelations are up-closed. But now we additionally have to require $Y = Y'$ to obtain $(Q\!:\!R)_{x,Z}$ as the intermediate set of the composition must contain *exactly* those states that are related to Z by R. This is due to the symmetric quotient/equivalence used by the alternative composition in contrast to the residual/implication used by Parikh's composition. For the alternative composition, the intermediate set of states Y is fixed to $R^{\mathsf{c}}(Z)$ for any given Z. This restriction for the choice of Y is also apparent in the general law $Q\!:\!R \subseteq Q\,;R$, which follows from properties (2) and (9) of Theorem 4.

We observe that there is yet another way of defining a composition $(Q :: R)$, namely by allowing $W \subseteq Z$ in the construction of Y' but then requiring $Y = Y'$. This is achieved by setting $Q :: R = Q : R\mathsf{S}$, which should be compared with property (2) of Theorem 4. However, it can be shown that this operation is neither associative nor $\subseteq$-isotone in its second argument nor $\subseteq$-antitone in its second argument. Specifically,

- $(\mathsf{E} :: \overline{\mathsf{E}}) :: \mathsf{T} = \mathsf{T}$ but $\mathsf{E} :: (\overline{\mathsf{E}} :: \mathsf{T}) = \mathsf{O}$,
- $(\overline{\mathsf{E}} :: \overline{\mathsf{E}}) :: \mathsf{T} = \mathsf{O}$ but $\overline{\mathsf{E}} :: (\overline{\mathsf{E}} :: \mathsf{T}) = \mathsf{T}$,
- $\overline{\mathsf{E}} :: \mathsf{O} = \mathsf{T}$ but $\overline{\mathsf{E}} :: \mathsf{T} = \mathsf{O}$,
- $\mathsf{E} :: \mathsf{O} = \mathsf{O}$ but $\mathsf{E} :: \mathsf{T} = \mathsf{T}$.

In conclusion, the various composition operations satisfy different algebraic properties, which are difficult to handle uniformly. In the remainder of this paper we focus on Parikh's composition.

5. Algebraic Structures for Investigating Multirelations

In this section we capture the properties of multirelations shown in Section 3 by seven algebraic structures, which are introduced in the following.

A *bounded join-semilattice* is an algebraic structure $(S, +, 0)$ satisfying for all $x, y, z \in S$ the associativity, commutativity, idempotence and neutrality axioms:

$$x + (y + z) = (x + y) + z \qquad x + y = y + x \qquad x + x = x \qquad 0 + x = x$$

The *semilattice order*, defined by $x \leq y$ if and only if $x + y = y$, for all $x, y \in S$, has the least element 0 and the least upper bound operation '+'. The operation '+' is $\leq$-isotone.

Next, a *bounded distributive lattice* $(S, +, \curlywedge, 0, \top)$ adds to a bounded join-semilattice a dual bounded meet-semilattice $(S, \curlywedge, \top)$ as well as distribution and absorption axioms, such that for all $x, y, z \in S$ the following equations hold:

$$\begin{aligned} x \curlywedge (y \curlywedge z) &= (x \curlywedge y) \curlywedge z & x + (y \curlywedge z) &= (x + y) \curlywedge (x + z) \\ x \curlywedge y &= y \curlywedge x & x \curlywedge (y + z) &= (x \curlywedge y) + (x \curlywedge z) \\ x \curlywedge x &= x & x + (x \curlywedge y) &= x \\ \top \curlywedge x &= x & x \curlywedge (x + y) &= x \end{aligned}$$

The semilattice order has the alternative characterisation that $x \leq y$ if and only if $x \curlywedge y = x$, for all $x, y \in S$. Moreover, $\top$ is the $\leq$-greatest element and '$\curlywedge$' is the $\leq$-greatest lower bound operation. The operation '$\curlywedge$' is $\leq$-isotone.

A *Boolean algebra* $(S, +, \curlywedge, ^{-}, 0, \top)$ expands a bounded distributive lattice by a complement operation $^{-}$ satisfying the following equations for all $x \in S$:

$$x \curlywedge \overline{x} = 0 \qquad\qquad x + \overline{x} = \top$$

A *pre-left semiring* $(S, +, \cdot, 0, 1)$ expands a bounded join-semilattice $(S, +, 0)$ with a binary operation '$\cdot$' and a constant 1 with the following axioms for all $x, y, z \in S$:

$$\begin{array}{cc} x = 1 \cdot x & (x \cdot y) + (x \cdot z) \leq x \cdot (y + z) \\ x \leq x \cdot 1 & (x \cdot z) + (y \cdot z) = (x + y) \cdot z \\ (x \cdot y) \cdot z \leq x \cdot (y \cdot z) & 0 = 0 \cdot x \end{array}$$

Note the inequalities in the left column. The operation '$\cdot$' is $\leq$-isotone. We often abbreviate a product $x \cdot y$ via juxtaposition to xy.

An *idempotent left semiring* (see [24]) is a pre-left semiring $(S, +, \cdot, 0, 1)$ whose reduct $(S, \cdot, 1)$ is a monoid, which is enforced by adding the axioms

$$x = x \cdot 1 \qquad\qquad (x \cdot y) \cdot z = x \cdot (y \cdot z),$$

for all $x, y, z \in S$. Idempotent semirings are rings in which the operation '$+$' is idempotent instead of having an inverse. Idempotent left semirings are idempotent semirings in which the operation '$\cdot$' is $\leq$-isotone instead of distributing over the operation '$+$' from the left and having the right zero 0. Pre-left semirings further weaken idempotent left semirings by requiring only one inequality of the associativity and right-neutral properties. This is because multirelations do not satisfy the other inequalities in general.

Combining the lattice and semiring operations, an *M0-algebra* is an algebraic structure $(S, +, \cdot, \curlywedge, 0, 1, \top)$ such that the reduct $(S, +, \curlywedge, 0, \top)$ is a bounded distributive lattice and the reduct $(S, +, \cdot, 0, 1)$ is a pre-left semiring. Finally, a *complemented M0-algebra* $(S, +, \cdot, \curlywedge, ^{-}, 0, 1, \top)$ has a Boolean algebra reduct $(S, +, \curlywedge, ^{-}, 0, \top)$ and a pre-left semiring reduct $(S, +, \cdot, 0, 1)$.

The algebraic results we will derive in the following sections apply to multirelations because of the following instances. The multirelations over a set A form a Boolean algebra $(A \leftrightarrow 2^A, \cup, \cap, ^{-}, \mathsf{O}, \mathsf{T})$. By Theorem 2 these multirelations also form a complemented M0-algebra $(A \leftrightarrow 2^A, \cup, ;, \cap, ^{-}, \mathsf{O}, \mathsf{E}, \mathsf{T})$ and the subset of up-closed multirelations forms both an M0-algebra $(A \overset{\mathsf{u}}{\leftrightarrow} 2^A, \cup, ;, \cap, \mathsf{O}, \mathsf{E}, \mathsf{T})$ and an idempotent left semiring $(A \overset{\mathsf{u}}{\leftrightarrow} 2^A, \cup, ;, \mathsf{O}, \mathsf{E})$. We refer to [28, 38] for further algebraic structures underlying up-closed multirelations and to [22] for placing them in a categorical setting. See also [27], where another kind of multirelational composition '$\cdot$' is introduced that gives rise to an M0-algebra. As shown in [18], this operation is not associative for general multirelations, but satisfies $(P \cdot Q) \cdot R \subseteq P \cdot (Q \cdot R)$ and $P = P \cdot 1$ for all P, Q and R, where $1 = \mathsf{I} \div \mathsf{E}$ is the singleton multirelation.

6. Reflexive-Transitive Closures of Multirelations

In relational computation models, the reflexive-transitive closure is frequently used to describe the semantics of iteration; for example, consider how while-loops are defined in Propositional Dynamic Logic [14]. In particular, the functions given by $f(x) = 1 + x \cdot y$ and $g(x) = 1 + y \cdot x$ and $h(x) = 1 + y + x \cdot x$ capture left recursion, right recursion and symmetric recursion, respectively. For relations, where 1 is the identity relation, the operation $+$ is union and the operation $\cdot$ is relational composition, the least fixpoints of f, g and h coincide with the reflexive-transitive closure of y. This is not clear for multirelational computation models because of the different properties of multirelational composition. In the present section we show how the least fixpoints of f, g and h are related in the case of multirelations.

To this end, we use left residuals and an appropriate algebraic structure. As proved in [17], multirelational composition has a left residual. If we define it by $R /\!\!/ Q := R/(\mathsf{E}\backslash Q)$, for all multirelations R and Q, we get

$$P ; Q \subseteq R \iff P(\mathsf{E}\backslash Q) \subseteq R \iff P \subseteq R/(\mathsf{E}\backslash Q) \iff P \subseteq R /\!\!/ Q,$$

for all multirelations P, Q and R.

A *residuated pre-left semiring* $(S,+,\cdot,/,0,1)$ expands a pre-left semiring $(S,+,\cdot,0,1)$ with a binary operation '$/$' satisfying the Galois connection

$$xy \leq z \iff x \leq z/y,$$

for all $x, y, z \in S$. It follows that the operation '$/$' is $\leq$-isotone in its first argument and $\leq$-antitone in its second argument. Moreover, we obtain the two properties $(x/y)y \leq x$ and $x/1 \leq x$, for all $x, y \in S$. As a consequence of the above calculation we get the following instance. The multirelations over a set A form a residuated pre-left semiring $(A \leftrightarrow 2^A, \cup, ;, /\!\!/, \mathsf{O}, \mathsf{E})$.

The following result considers the $\leq$-isotone functions f, g and h. The $\leq$-least prefixpoint μf of the function f is axiomatised using its unfold and induction properties, that is, $f(\mu f) \leq \mu f$ and that $f(x) \leq x$ implies $\mu f \leq x$, for all $x \in S$. Because f is $\leq$-isotone it follows that $f(\mu f) = \mu f$, whence μf is also the $\leq$-least fixpoint of f. Similar axioms are assumed for μg and μh.

Theorem 5. *Let S be a residuated pre-left semiring and let $y \in S$. Depending on y, let f, g and h be functions on S defined by*

$$f(x) = 1 + x \cdot y \qquad\qquad g(x) = 1 + y \cdot x \qquad\qquad h(x) = 1 + y + x \cdot x,$$

for all $x \in S$. Assume that μf, μg and μh exist. Then we have $\mu f \leq \mu g = \mu h$.

Proof. We first show $\mu f \leq \mu g$. Semi-associativity of composition, the Galois property of the left residual and the prefixpoint property of μg imply

$$(y \cdot (\mu g/y)) \cdot y \leq y \cdot ((\mu g/y) \cdot y) \leq y \cdot \mu g \leq 1 + y \cdot \mu g \leq \mu g.$$

Hence, we get $y \cdot (\mu g/y) \leq \mu g/y$. Moreover, $1 \leq 1 + y \cdot \mu g \leq \mu g$ holds, whence semi-neutrality of composition gives

$$1 \cdot y = y \leq y \cdot 1 \leq 1 + y \cdot \mu g \leq \mu g.$$

So, $1 \leq \mu g/y$ and, together, we have

$$g(\mu g/y) = 1 + y \cdot (\mu g/y) \leq \mu g/y.$$

From this we obtain $\mu g \leq \mu g/y$ by the least prefixpoint property of μg. Hence

$$f(\mu g) = 1 + \mu g \cdot y \leq \mu g$$

and, therefore, $\mu f \leq \mu g$ follows by the least prefixpoint property of μf.

We next show $\mu g \leq \mu h$. This part does not use residuals. From the least prefixpoint property of μh we get $y \leq 1 + y + \mu h \cdot \mu h = h(\mu h) \leq \mu h$; hence

$$g(\mu h) = 1 + y \cdot \mu h \leq 1 + y + \mu h \cdot \mu h = h(\mu h) \leq \mu h$$

by the prefixpoint property of μh. Therefore, we arrive at $\mu g \leq \mu h$ by the least prefixpoint property of μg.

We finally show $\mu h \leq \mu g$ following the argument of [6, Satz 10.1.5], which is for homogeneous relations. Semi-associativity of composition, a property of the left residual and the unfold property of μg imply:

$$\begin{aligned} g(\mu g/\mu g) \cdot \mu g &= (1 + y \cdot (\mu g/\mu g)) \cdot \mu g = 1 \cdot \mu g + (y \cdot (\mu g/\mu g)) \cdot \mu g \\ &\leq \mu g + y \cdot ((\mu g/\mu g) \cdot \mu g) \leq \mu g + 1 + y \cdot \mu g = \mu g + g(\mu g) = \mu g \end{aligned}$$

As a consequence we obtain $g(\mu g/\mu g) \leq \mu g/\mu g$ and this leads to $\mu g \leq \mu g/\mu g$ by the least prefixpoint property of μg, whence $\mu g \cdot \mu g \leq \mu g$. With $1 \leq \mu g$ and $y \leq \mu g$ shown above, it follows that

$$h(\mu g) = 1 + y + \mu g \cdot \mu g \leq \mu g.$$

Therefore we have $\mu h \leq \mu g$ by the least prefixpoint property of μh. $\square$

R or x is ...	if and only if	algebraically
total	$R;\mathsf{T} = \mathsf{T}$	$x\top = \top$
co-total	$R;\mathsf{O} = \mathsf{O}$	$x0 = 0$
transitive	$R;R \subseteq R$	$xx \leq x$
dense	$R \subseteq R;R$	$x \leq xx$
reflexive	$\mathsf{E} \subseteq R$	$1 \leq x$
co-reflexive	$R \subseteq \mathsf{E}$	$x \leq 1$
idempotent	$R;R = R$	$xx = x$
up-closed	$R;\mathsf{E} = R$	$x1 = x$
down-closed	$\overline{R};\mathsf{E} = \overline{R}$	$\overline{x}1 = \overline{x}$
$\cup$-distributive	$R;(P \cup Q) = R;P \cup R;Q$	$x(y + z) = xy + xz$
$\cap$-distributive	$R;(P \cap Q) = R;P \cap R;Q$	$x(y \curlywedge z) = xy \curlywedge xz$
a contact	$R;R \cup \mathsf{E} = R$	$xx + 1 = x$
a kernel	$R;R \cap \mathsf{E} = R;\mathsf{E}$	$xx \curlywedge 1 = x1$
a test	$R;\mathsf{T} \cap \mathsf{E} = R$	$x\top \curlywedge 1 = x$
a co-test	$R;\mathsf{O} \cup \mathsf{E} = R$	$x0 + 1 = x$
a zero-vector	$R \subseteq R;\mathsf{O}$	$x \leq x0$
a one-vector	$R;\mathsf{O} \subseteq R$	$x0 \leq x$
a vector	$R;\mathsf{T} = R$	$x\top = x$

Figure 1: Fundamental properties

For up-closed multirelations the equality $\mu g = \mu h$ is shown in [37]. Furthermore, for finitary up-closed multirelations $\bigcup_{n\in\mathbb{N}} g^n(\mathsf{O}) \subseteq \mu h$ is shown in [20] and $\bigcup_{n\in\mathbb{N}} g^n(\mathsf{O}) = \mu g$ is shown in [17].

While reflexive-transitive closures have originally been considered for relations, they can be generalised to other structures that support notions of reflexivity and transitivity. We will see in Section 7 that this is the case for multirelations. In particular, the proof of Theorem 5 implies that μh is the $\leq$-least element above y that is reflexive and transitive in this sense. Moreover, the operation that maps y to μh is a closure operation.

We proved Theorem 5 also in Isabelle/HOL using its integrated automated theorem provers and SMT solvers, which are described in [11, 26]. The same holds for the theorems we will present in Sections 7 and 8, that is, Theorem 6 to Theorem 10. We therefore omit their proofs, which are given in the Isabelle/HOL theory files available at http://www.csse.canterbury.ac.nz/walter.guttmann/algebra/.

7. Properties of Multirelations

A number of properties of multirelations were used in previous work for modelling games, protocols, computations, contact, closure and topology; for example, see [1, 5, 23, 25, 29]. Algebraic definitions of these and other properties are summarised in Figure 1. Its second column states the property in terms of relations and the third column gives the corresponding definition in (complemented) M0-algebras. The distributivity properties universally quantify over the multirelations P, Q and the elements y, z of the M0-algebra, respectively.

For up-closed multirelations several of the properties listed in Figure 1 are dual to each other, that is, can be obtained by applying the multirelational dual operation. This does not hold for general multirelations: for example, the conjunction of reflexive and transitive implies up-closed, but the conjunction of their duals co-reflexive and dense does not imply up-closed, which is self-dual.

In this section we investigate the connections between the properties in Figure 1 using the algebraic structure of multirelations. While many results can be derived in M0-algebras, additional axioms are needed to prove some others, leading to the following new algebraic structure. An *M1-algebra* is an M0-algebra $(S, +, \cdot, \curlywedge, 0, 1, \top)$ satisfying the axioms

$$\top = \top x \qquad x(yz) = (x(y1))z \qquad xz \curlywedge yz = (x1 \curlywedge y1)z,$$

for all $x, y, z \in S$. An equivalent structure is obtained if just '$\leq$' is assumed instead of equality in each axiom. If all elements are up-closed, that is, $x1 = x$ holds for all $x \in S$, the last two axioms collapse to associativity of the operation '$\cdot$' and right-distributivity of '$\cdot$' over the operation '$\curlywedge$'. This shows how to obtain weaker axioms which hold for arbitrary multirelations.

7.1. Zero-vectors and one-vectors

Observe that for relations and relational composition $R\mathsf{O} \subseteq R = R\mathsf{I} \subseteq R\mathsf{T}$ holds. Not all of these inclusions generalise to multirelations; for these we only have $x \leq x1 \leq x\top$ and $x0 \leq x1 \leq x\top$. In general, neither does $x1 \leq x$ hold nor is x comparable to $x0$. Requiring some of these inequalities gives various subclasses of multirelations characterised by the properties in Figure 1. In particular, we obtain:

- zero-vectors by requiring $x \leq x0$ or any of the following equivalent conditions:

$$\begin{array}{rl} x1 = x0 & \forall y : xy = x0 \\ x1 \leq x0 & \forall y : xy \leq x0 \\ x\top = x0 & \forall y : x\top = xy \\ x\top \leq x0 & \forall y : x\top \leq xy \\ x\top = x1 & \forall y : x1 = xy \\ x\top \leq x1 & \forall y, z : xy = xz \end{array}$$

- one-vectors by requiring $x0 \leq x$;
- up-closed multirelations by requiring $x1 \leq x$ or, equivalently, $x1 = x$;
- vectors by requiring $x\top \leq x$ or, equivalently, either $x\top = x$ or $x0 = x$.

It follows that every vector is up-closed, every up-closed multirelation is a one-vector and every vector is a zero-vector. Moreover, a multirelation is a vector if and only if it is both a zero-vector and a one-vector. The names of these properties are suggested by the interpretation of multirelations as binary matrices:

- A multirelation $R : A \leftrightarrow 2^B$ is a zero-vector if and only if $\neg R_{x,\emptyset}$ implies $\neg R_{x,Y}$, for all $x \in A$ and $Y \in 2^B$. Thus if the entry $R_{x,\emptyset}$ is 0, then all entries in row x must be 0. No restriction is imposed if $R_{x,\emptyset}$ is 1.
- R is a one-vector if and only if $R_{x,\emptyset}$ implies $R_{x,Y}$, for all $x \in A$ and $Y \in 2^B$. Thus if the entry $R_{x,\emptyset}$ is 1, then all entries in row x must be 1. No restriction is imposed if $R_{x,\emptyset}$ is 0.

Therefore, a multirelation is a vector if and only if it is up-closed and a zero-vector. To make an equivalent statement for one-vectors we need a new concept discussed in the following subsection.

7.2. Down-closed multirelations and the complement operation

A multirelation $R : A \leftrightarrow 2^B$ is *down-closed* if each element of A that is related to a set Y is also related to all subsets of Y or, formally, if

$$R_{x,Y} \wedge Z \subseteq Y \implies R_{x,Z}$$

for all $x \in A$ and $Y, Z \in 2^B$. Using this property, we can characterise vectors as those multirelations which are down-closed and one-vectors.

However, it is not known how to express the property of being down-closed using the operations of M0-algebras, that is, union, intersection, composition and the constants 0, 1 and $\top$. Certainly, being down-closed cannot be expressed using universally quantified conjunctions of inequalities of the form $s \leq t$ for $s, t \in \{x, x0, x1, x\top\}$ as all of these yield either vectors, zero-vectors, one-vectors or up-closed multirelations. This is one of the motivations for adding a complement operation, based on the observation that the lattice structure of multirelations is in fact a Boolean algebra. Because a multirelation x is down-closed if and only if its complement $\overline{x}$ is up-closed, we obtain the characterisation $\overline{x}1 = \overline{x}$ for down-closed multirelations. To investigate this property and to relate it to other properties we use the following algebraic structure.

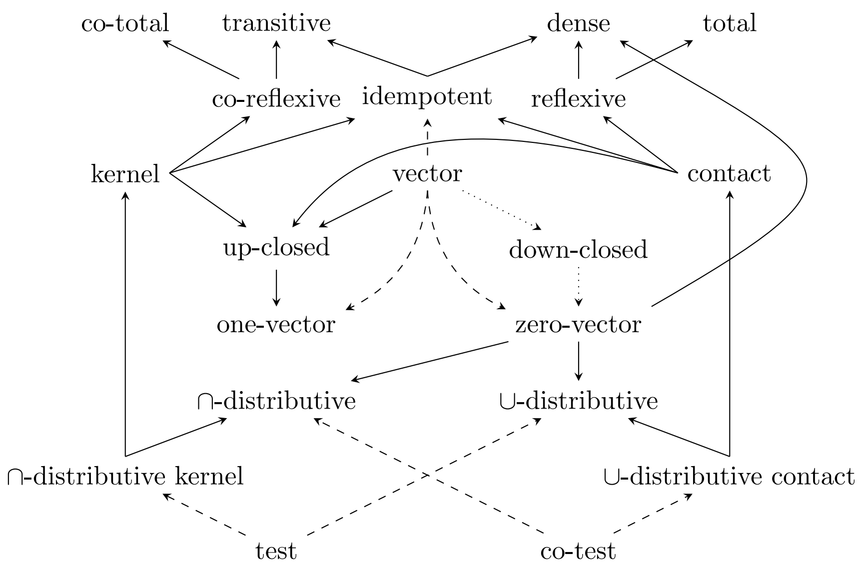

Figure 2: Relationships between the fundamental properties

A *complemented M1-algebra* is a complemented M0-algebra $(S, +, \cdot, \curlywedge, ^{-}, 0, 1, \top)$ whose complement-free reduct $(S, +, \cdot, \curlywedge, 0, 1, \top)$ is an M1-algebra satisfying the following additional axioms for all $x, y \in S$:

$$x\top \curlywedge y \leq (x \curlywedge y\top)\top \qquad\qquad (x0 \curlywedge y)0 \leq (x \curlywedge y)0$$

It can be shown that the multirelations over a set A satisfy these axioms.

Using the complement operation, we obtain the following equivalent characterisations of zero-vectors, one-vectors and vectors. An element $x \in S$ is a zero-vector if and only if $\overline{x}0 \leq \overline{x}$. Next, x is a one-vector if and only if $\overline{x} \leq \overline{x}0$ or, equivalently, $\overline{x}y = \overline{x}z$ for all $y, z \in S$. Finally, x is a vector if and only if $\overline{x}0 = \overline{x}$ or, equivalently, $\overline{x}\top = \overline{x}$. The following theorem summarises our results about relationships between the properties in Figure 1.

Theorem 6. *The implications shown in Figure 2 drawn as continuous / dashed / dotted arrows hold in M0-algebras / M1-algebras / complemented M1-algebras. Furthermore, arrows originating in the same point indicate that the property is equivalent to the conjunction of the targets.*

Furthermore, we obtain the following properties in complemented M1-algebras. In particular, note that property (4) is a special case of the Schröder equivalences which hold for relations. Also similarly to relations, property (6) shows that the intersection with a vector can be exported from the first argument of a composition.

Theorem 7. *Let S be a complemented M1-algebra and let $x, y, z \in S$. Then we have:*

(1) $\overline{x0} \leq \overline{x}0$.
(2) $\overline{x}$ *is a zero-vector if* x *is transitive or up-closed.*
(3) $\overline{x\top}\top \leq \overline{x}$.
(4) $x\top \leq y \iff \overline{y}\top \leq \overline{x}$.
(5) $\overline{x\top}\top = \overline{x\top}1 = \overline{x\top}$.
(6) $x\top \curlywedge yz = (x\top \curlywedge y)z$.
(7) $x0 \curlywedge yz = (x0 \curlywedge y)z = (x0 \curlywedge y1)z$.
(8) $x0 \curlywedge y0 = (x \curlywedge y)0 = (x0 \curlywedge y)0 = (x0 \curlywedge y0)0 = (x0 \curlywedge y1)0 = (x1 \curlywedge y1)0 = (x1 \curlywedge y)0$.

8. Closure Properties of Multirelational Operations

It is known that up-closed multirelations are closed under the multirelational operations we have introduced in Section 3. In this section we systematically investigate the closure properties for certain classes of multirelations, which are given by the properties presented in Figure 1. For dealing with the dual operation we need additional axioms, which lead to the expansions of M0-algebras we will introduce in this section.

First, an *M2-algebra* $(S, +, \cdot, \curlywedge, ^{\mathsf{d}}, 0, 1, \top)$ is an M0-algebra $(S, +, \cdot, \curlywedge, 0, 1, \top)$ expanded with a unary dual operation '$^{\mathsf{d}}$' satisfying the axioms

$$(xy)^{\mathsf{d}} = (x1)^{\mathsf{d}} y^{\mathsf{d}} \qquad (x+y)^{\mathsf{d}} = x^{\mathsf{d}} \curlywedge y^{\mathsf{d}} \qquad x^{\mathsf{dd}} = x \qquad 1^{\mathsf{d}} = 1,$$

for all $x, y \in S$. Note again how distributivity of the operation '$^{\mathsf{d}}$' over the operation '$\cdot$', which holds for up-closed multirelations, is weakened by replacing x with $x1$.

A *complemented M2-algebra* $(S, +, \cdot, \curlywedge, ^{-}, ^{\mathsf{d}}, 0, 1, \top)$ has a Boolean algebra reduct $(S, +, \curlywedge, ^{-}, 0, \top)$ and an M2-algebra reduct $(S, +, \cdot, \curlywedge, ^{\mathsf{d}}, 0, 1, \top)$. The above axioms imply the additional axioms of M1-algebras. Thus, we obtain the following result.

Theorem 8. *All M2-algebras are M1-algebras.*

For reasoning about up-closed multirelations we use that the operation '$^{\mathsf{d}}$' distributes over the operation '$\cdot$'. As a further expansion of M0-algebras, therefore, an *M3-algebra* $(S, +, \cdot, \curlywedge, ^{\mathsf{d}}, 0, 1, \top)$ is an M0-algebra $(S, +, \cdot, \curlywedge, 0, 1, \top)$ expanded with a unary dual operation '$^{\mathsf{d}}$' satisfying the axioms

$$(xy)^{\mathsf{d}} = x^{\mathsf{d}} y^{\mathsf{d}} \qquad (x+y)^{\mathsf{d}} = x^{\mathsf{d}} \curlywedge y^{\mathsf{d}} \qquad x^{\mathsf{dd}} = x \qquad 1^{\mathsf{d}} = 1,$$

for all $x, y \in S$. These axioms imply the axioms of M2-algebras. Moreover, we obtain that the operation '$\cdot$' is associative with right-neutral element 1, that is, the idempotent left semiring structure.

Theorem 9. *All M3-algebras are M2-algebras and idempotent left semirings.*

The algebraic results obtained so far apply to multirelations due to the following instances. By Theorem 3, the multirelations over a set A form a complemented M2-algebra $(A \leftrightarrow 2^A, \cup, ;, \cap, ^{-}, ^{\mathsf{d}}, \mathsf{O}, \mathsf{E}, \mathsf{T})$ and the up-closed multirelations over A form an M3-algebra $(A \overset{\mathrm{u}}{\leftrightarrow} 2^A, \cup, ;, \cap, ^{\mathsf{d}}, \mathsf{O}, \mathsf{E}, \mathsf{T})$. Note that up-closed multirelations are not closed under the set complement operation. Indeed, assuming the existence of complements in an M3-algebra would imply that all elements are vectors. The next theorem summarises the closure properties of multirelations.

Theorem 10. *Figure 3 shows which properties in Figure 1 hold for the multirelational constants and with respect to which operations these properties are closed. There an entry* ■ *(□) means that the property is closed under the respective operation in complemented M2-algebras (M3-algebras). An entry* ▼/▲ *(▽/△) in the column of operation f means that if x satisfies the property then $f(x)$ satisfies the property below/above in complemented M2-algebras (M3-algebras). An entry $-$ means that the property is not closed under the respective operation even for up-closed multirelations.*

To give an example, the dual of a co-total multirelation is total and the dual of an up-closed total multirelation is co-total. Another consequence of the closure properties are sub-algebras. For example, the set of co-total multirelations forms a pre-left semiring and so does the set of co-reflexive multirelations.

Counterexamples witness that the claims in Figure 3 cannot be strengthened. They are shown in Figures 4, 5, 6 and 7 as Boolean matrices (where a grey square denotes a 1-entry and a white square denotes a 0-entry). Most counterexamples have been found using a Haskell program which performs an exhaustive search. For $\cup$- and $\cap$-distributivity of up-closed multirelations we use the alternative characterisation provided by Aumann contacts given in Section 9. An infinite counterexample which shows that $\cup$-distributivity is not preserved by composition is given in Theorem 21 in Section 10.

	O	E	T	∪	∩	;	d	—
total	—	■	■	■	□	□	▽	—
co-total	■	■	—	■	■	■	▲	▲
transitive	■	■	■	—	■	—	▼	▼
dense	■	■	■	■	—	—	△	□
reflexive	—	■	■	■	■	■	▼	—
co-reflexive	■	■	—	■	■	■	▲	—
idempotent	■	■	■	—	—	—	□	—
up-closed	■	■	■	■	■	■	■	▼
down-closed	■	—	■	■	■	■	■	▲
∪-distributive	■	■	■	■	—	□	▽	□
∩-distributive	■	■	■	—	□	□	△	□
a contact	—	■	■	—	■	—	▼	—
a kernel	■	■	—	■	—	—	▲	—
a ∪-distributive contact	—	■	■	—	—	—	▼	—
a ∩-distributive kernel	■	■	—	—	—	—	▲	—
a test	■	■	—	■	■	■	▼	—
a co-test	—	■	■	■	■	■	▲	—
a zero-vector	■	—	■	■	■	■	□	▼
a one-vector	■	■	■	■	■	■	□	▲
a vector	■	—	■	■	■	■	■	■

Figure 3: Closure properties of multirelations

Note that M2-algebras are not complete for multirelations. The counterexample generator Nitpick, which is described in [12], finds a counterexample showing that $x\top \curlywedge yz \le (x\top \curlywedge y)z$ does not follow in M2-algebras. However, this property holds for multirelations since by Theorem 7 it follows in complemented M1-algebras.

Neither are M3-algebras complete for up-closed multirelations. Nitpick shows that $x\top \curlywedge x^{\mathsf{d}}0 = 0$ does not follow in M3-algebras, although it holds for up-closed multirelations. To see this, note that this equation is an axiom of 'algebras of monotonic Boolean transformers' of [28] or consider the following proof. Let R be an up-closed multirelation. Then we have $R(\mathsf{E}\backslash\mathsf{E}) = R\,;\mathsf{E} = R$. By a Schröder equivalence we get $R^{\mathsf{c}}\overline{R} \subseteq \mathsf{E}^{\mathsf{c}}\overline{\mathsf{E}} \subseteq \mathsf{T}\overline{\mathsf{E}}$. Hence, $\mathsf{T}R^{\mathsf{c}}\overline{R}\mathsf{C} \subseteq \mathsf{T}\overline{\mathsf{E}}\mathsf{C} = \mathsf{T}\mathsf{E}$. Another Schröder equivalence gives $\overline{R}\mathsf{C}\overline{\mathsf{E}^{\mathsf{c}}\mathsf{T}} \subseteq \overline{R\mathsf{T}}$. So,

$$R;\mathsf{T} \cap R^{\mathsf{d}};\mathsf{O} = R\mathsf{T} \cap R^{\mathsf{d}}(\mathsf{E}\backslash\mathsf{O}) = R\mathsf{T} \cap \overline{R}\mathsf{C}\overline{\mathsf{E}^{\mathsf{c}}\mathsf{T}} \subseteq R\mathsf{T} \cap \overline{R\mathsf{T}} = \mathsf{O}$$

shows the desired result.

9. Aumann Contacts and Multirelational Properties

In [1, 2, 3, 4] Aumann investigated certain laws for modelling the notion of a contact in topology. Translated into the language of multirelations, he considered for a multirelation $R : A \leftrightarrow 2^A$ the following five axioms:

$$\begin{array}{ll} (K_0) & \neg\exists x \in A : R_{x,\emptyset} \\ (K_1) & \forall x \in A : R_{x,\{x\}} \\ (K_2) & \forall x \in A : \forall Y, Z \in 2^A : R_{x,Y} \wedge Y \subseteq Z \Rightarrow R_{x,Z} \\ (K_3) & \forall x \in A : \forall Y, Z \in 2^A : R_{x,Y} \wedge (\forall y \in Y : R_{y,Z}) \Rightarrow R_{x,Z} \\ (K_4) & \forall x \in A : \forall Y, Z \in 2^A : R_{x,Y \cup Z} \Leftrightarrow R_{x,Y} \vee R_{x,Z} \end{array}$$

Aumann called multirelations satisfying the formulas (K_1) to (K_3) 'contact relations' and multirelations satisfying the formulas (K_0) to (K_4) 'topological contact relations'. In this section we give multirelation-algebraic characterisations of these logical formulas. See [33] for the relation-algebraic treatment of a correspondence between contact relations and closure operations. Axioms (K_0), (K_2) and (K_4) generalise to

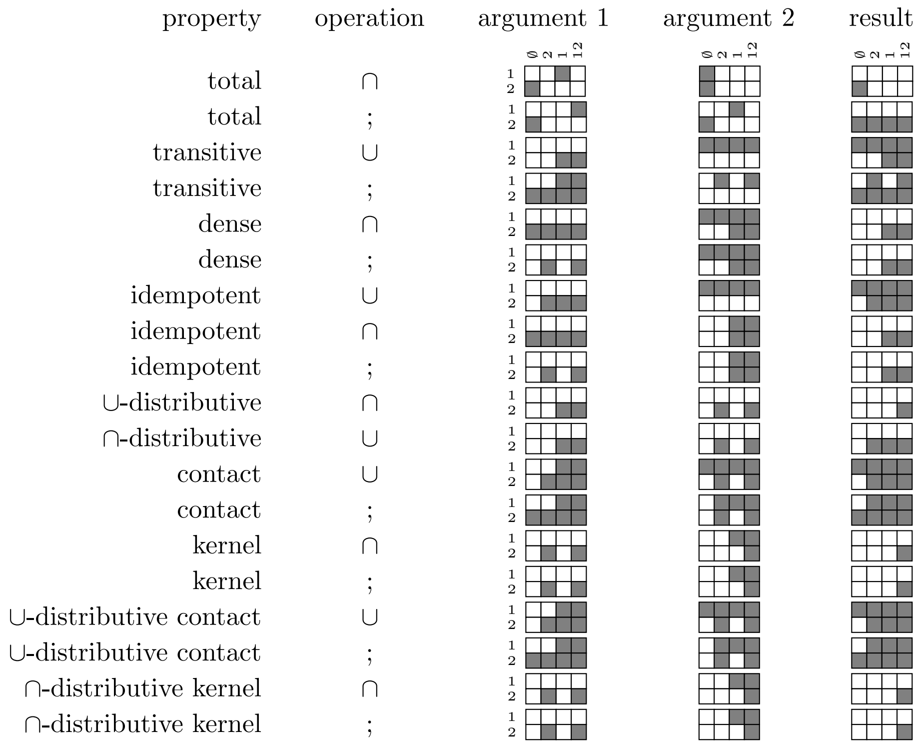

Figure 4: Counterexamples generated by a Haskell program

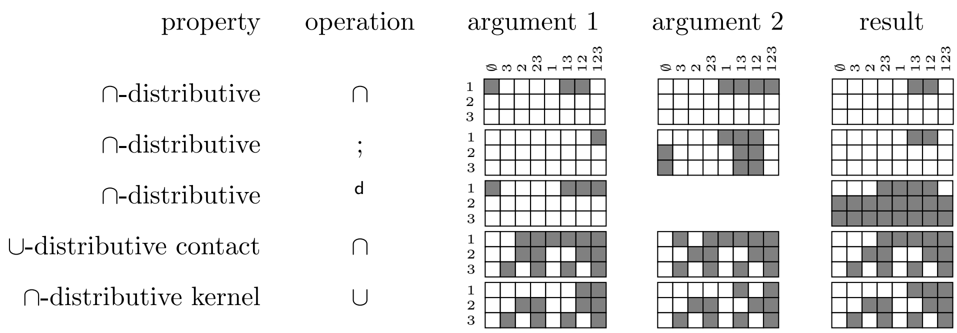

Figure 5: Manually generated counterexamples

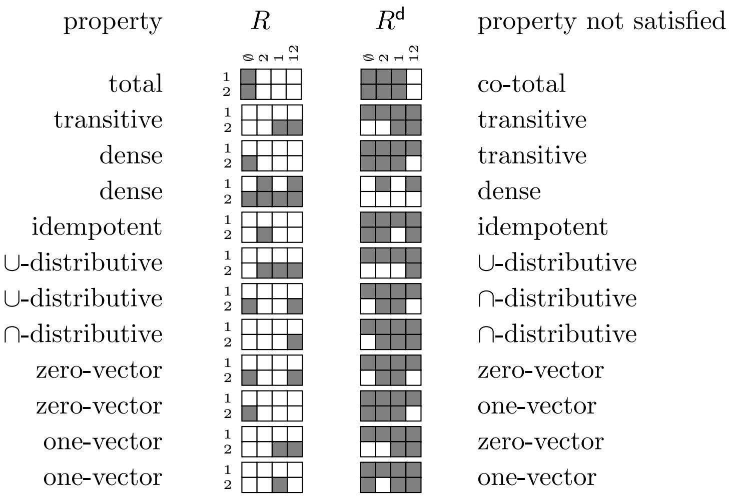

Figure 6: Counterexamples for the dual operation generated by a Haskell program

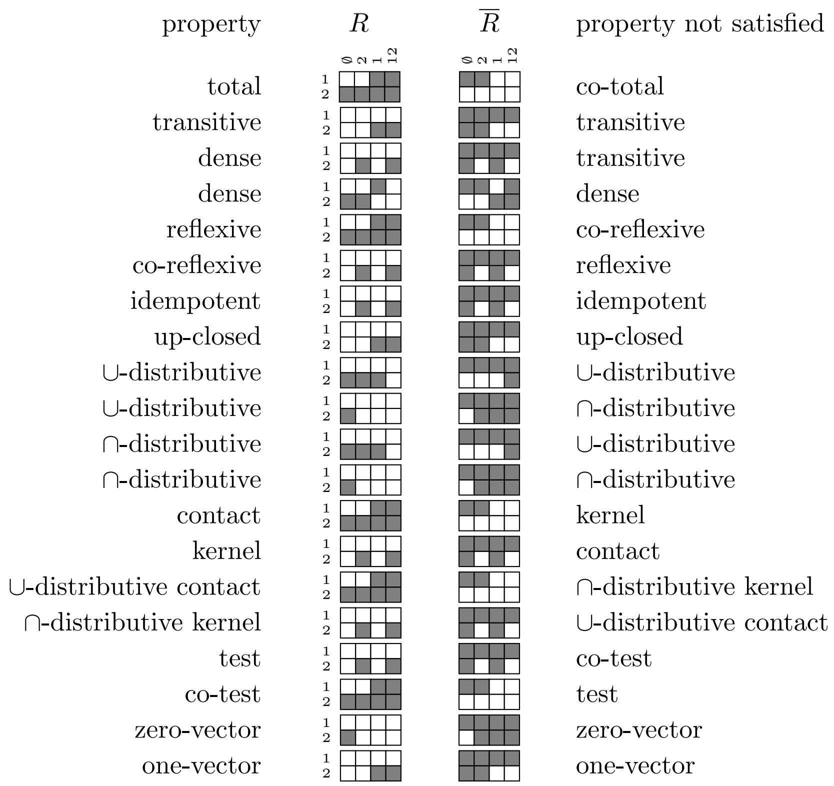

Figure 7: Counterexamples for the complement operation generated by a Haskell program

multirelations of type $A \leftrightarrow 2^B$ in a straight-forward way. The following result gives the property corresponding to K_0.

Theorem 11. *A multirelation satisfies* (K_0) *if and only if it is co-total.*

Proof. Axiom (K_0) applied to a multirelation $R : A \leftrightarrow 2^B$ elaborates as follows:

$$\begin{aligned}
\neg\exists x \in A : R_{x,\emptyset} &\iff \forall x \in A : \neg R_{x,\emptyset} \\
&\iff \forall x \in A : \forall X \in 2^B : R_{x,X} \Rightarrow X \neq \emptyset \\
&\iff \forall x \in A : \forall X \in 2^B : R_{x,X} \Rightarrow \exists y \in B : y \in X \\
&\iff \forall x \in A : \forall X \in 2^B : R_{x,X} \Rightarrow \exists y \in B : \mathsf{T}_{x,y} \wedge \mathsf{E}_{y,X} \\
&\iff \forall x \in A : \forall X \in 2^B : R_{x,X} \Rightarrow (\mathsf{TE})_{x,X} \\
&\iff R \subseteq \mathsf{TE} \\
&\iff \mathsf{T}R \subseteq \mathsf{TE} \\
&\iff R\overline{\mathsf{TE}}^{\mathsf{c}} \subseteq \mathsf{O} \\
&\iff R(\mathsf{E}\backslash\mathsf{O}) \subseteq \mathsf{O} \\
&\iff R\,;\mathsf{O} \subseteq \mathsf{O}
\end{aligned}$$

Hence, the characterisation in Figure 1 shows the claim. □

The forward implication of this theorem is stated in [30], where such multirelations are called ‘total’. We call the above property ‘co-total’ to keep the standard use of ‘total‘ known from relations and functions. Namely, the calculation

$$R\,;\mathsf{T} = R(\mathsf{E}\backslash\mathsf{T}) = R\overline{\mathsf{E}^{\mathsf{c}}\overline{\overline{\mathsf{T}}}} = R\overline{\mathsf{E}^{\mathsf{c}}\mathsf{O}} = R\overline{\mathsf{O}} = R\mathsf{T}$$

implies that the multirelation-algebraic property $R\,;\mathsf{T} = \mathsf{T}$ is equivalent to the relation-algebraic property of totality $R\mathsf{T} = \mathsf{T}$. In [29] multirelations R satisfying the property $R\,;\mathsf{T} = \mathsf{T}$ are called ‘proper’. Next, we investigate axiom (K_1) and relate it to a property in Figure 1.

Theorem 12. *Every reflexive multirelation satisfies* (K_1)*. An up-closed multirelation satisfies* (K_1) *if and only if it is reflexive.*

Proof. Axiom (K_1) applied to a multirelation $R : A \leftrightarrow 2^A$ elaborates as follows:

$$\begin{aligned}
\forall x \in A : R_{x,\{x\}} &\iff \forall x \in A : \forall X \in 2^A : \{x\} = X \Rightarrow R_{x,X} \\
&\Longleftarrow \forall x \in A : \forall X \in 2^A : \{x\} \subseteq X \Rightarrow R_{x,X} \\
&\iff \forall x \in A : \forall X \in 2^A : x \in X \Rightarrow R_{x,X} \\
&\iff \forall x \in A : \forall X \in 2^A : \mathsf{E}_{x,X} \Rightarrow R_{x,X} \\
&\iff \mathsf{E} \subseteq R
\end{aligned}$$

Again Figure 1 shows the first claim. If R is up-closed, then the reverse implication holds since $R_{x,\{x\}}$ and $\{x\} \subseteq X$ imply $R_{x,X}$. □

Axiom (K_2) is the logical characterisation of R being an up-closed multirelation. For this property, the relation-algebraic characterisation $R = R\mathsf{S}$ is shown in [21, Theorem 6] and the multirelation-algebraic characterisation $R\,;\mathsf{E} = R$ is shown in [21, Theorem 7.1]. With respect to axiom (K_3), we have the following correspondence.

Theorem 13. *A multirelation satisfies* (K_3) *if and only if it is transitive.*

Proof. Axiom (K_3) applied to a multirelation $R : A \leftrightarrow 2^A$ elaborates as follows:

$$\begin{aligned}
R\,;R \subseteq R &\iff R(\mathsf{E}\backslash R) \subseteq R \\
&\iff \forall x \in A : \forall Z \in 2^A : (R(\mathsf{E}\backslash R))_{x,Z} \Rightarrow R_{x,Z} \\
&\iff \forall x \in A : \forall Z \in 2^A : (\exists Y \in 2^A : R_{x,Y} \wedge (\mathsf{E}\backslash R)_{Y,Z}) \Rightarrow R_{x,Z} \\
&\iff \forall x \in A : \forall Y, Z \in 2^A : R_{x,Y} \wedge (\mathsf{E}\backslash R)_{Y,Z} \Rightarrow R_{x,Z} \\
&\iff \forall x \in A : \forall Y, Z \in 2^A : R_{x,Y} \wedge (\forall y \in A : \mathsf{E}_{y,Y} \Rightarrow R_{y,Z}) \Rightarrow R_{x,Z} \\
&\iff \forall x \in A : \forall Y, Z \in 2^A : R_{x,Y} \wedge (\forall y \in A : y \in Y \Rightarrow R_{y,Z}) \Rightarrow R_{x,Z} \\
&\iff \forall x \in A : \forall Y, Z \in 2^A : R_{x,Y} \wedge (\forall y \in Y : R_{y,Z}) \Rightarrow R_{x,Z}
\end{aligned}$$

Again Figure 1 shows the claim. □

Taken together, the axioms (K_1) to (K_3) of Aumann are equivalent to multirelations being reflexive, up-closed and transitive (or even idempotent, since reflexive implies dense). Finally, we investigate axiom (K_4). Here we obtain the following result. Its proof is given in Section 10 as a consequence of a more general characterisation.

Theorem 14. *Multirelations satisfying (K_4) are $\cup$-distributive. An up-closed multirelation satisfies (K_4) if and only if it is $\cup$-distributive.*

Extended to arbitrary non-empty unions, axiom (K_4) is called 'additive' in [29], which also states that additive up-closed multirelations are $\cup$-distributive. Finally we consider the dual property of axiom (K_4), that is, the following logical formula for a given multirelation $R : A \leftrightarrow 2^B$:

$$(K_4') \quad \forall x \in A : \forall Y, Z \in 2^B : R_{x,Y} \wedge R_{x,Z} \Leftrightarrow R_{x,Y \cap Z}$$

Extended to arbitrary non-empty unions, this is called 'multiplicative' in [30], which also states that multiplicative up-closed multirelations are $\cap$-distributive. Also the following result is proved as a corollary of a more general characterisation in Section 10.

Theorem 15. *Multirelations satisfying (K_4') are $\cap$-distributive. An up-closed multirelation satisfies (K_4') if and only if it is $\cap$-distributive.*

10. Characterisation of ∩-distributive and ∪-distributive multirelations

Extending Theorems 14 and 15, the characterisations of $\cap$-distributivity and $\cup$-distributivity of multirelations provided in this section work for arbitrary multirelations. They involve fewer variables than the original definitions, which makes them easier to establish and faster to refute due to the reduced search space. We start with the characterisation of $\cap$-distributive multirelations.

Theorem 16. *A multirelation $R : A \leftrightarrow 2^B$ is $\cap$-distributive if and only if*

$$\forall x \in A : \forall Y, Z \in 2^B : R_{x,Y} \wedge R_{x,Z} \Rightarrow \exists W \in 2^B : R_{x,W} \wedge W \subseteq Y \cap Z \tag{1}$$

Proof. To show the forward implication, let R be $\cap$-distributive. Let $x \in A$ and $Y, Z \in 2^B$ be such that $R_{x,Y}$ and $R_{x,Z}$. We define the multirelations $P, Q : B \leftrightarrow 2^B$ by $P_{w,V}$ if and only if $w \in Y$ and $Q_{w,V}$ if and only if $w \in Z$, for all $w \in B$ and $V \in 2^B$. Then (1) follows since, for any $V \in 2^B$, we have:

$$\begin{aligned}
R_{x,Y} \wedge R_{x,Z} &\implies (\exists W \in 2^B : R_{x,W} \wedge W \subseteq Y) \wedge (\exists W \in 2^B : R_{x,W} \wedge W \subseteq Z) \\
&\iff (\exists W \in 2^B : R_{x,W} \wedge \forall w \in W : w \in Y) \wedge (\exists W \in 2^B : R_{x,W} \wedge \forall w \in W : w \in Z) \\
&\iff (\exists W \in 2^B : R_{x,W} \wedge \forall w \in W : P_{w,V}) \wedge (\exists W \in 2^B : R_{x,W} \wedge \forall w \in W : Q_{w,V}) \\
&\iff (R;P)_{x,V} \wedge (R;Q)_{x,V} \\
&\iff (R;P \cap R;Q)_{x,V} \\
&\iff (R;(P \cap Q))_{x,V} \\
&\iff \exists W \in 2^B : R_{x,W} \wedge \forall w \in W : (P \cap Q)_{w,V} \\
&\iff \exists W \in 2^B : R_{x,W} \wedge \forall w \in W : P_{w,V} \wedge Q_{w,V} \\
&\iff \exists W \in 2^B : R_{x,W} \wedge \forall w \in W : w \in Y \wedge w \in Z \\
&\iff \exists W \in 2^B : R_{x,W} \wedge W \subseteq Y \cap Z
\end{aligned}$$

To show the backward implication, assume that (1) holds. Let $P, Q : B \leftrightarrow 2^C$ be multirelations and $x \in A$ and $V \in 2^C$ such that $(R;P \cap R;Q)_{x,V}$. Then we have:

$$(\exists Y \in 2^B : R_{x,Y} \wedge \forall y \in Y : P_{y,V}) \wedge (\exists Z \in 2^B : R_{x,Z} \wedge \forall z \in Z : Q_{z,V})$$

By (1), there exists a set $W \subseteq Y \cap Z$ such that $R_{x,W}$. We show $(P \cap Q)_{w,V}$ for each $w \in W$. This follows since $w \in W \subseteq Y$ implies $P_{w,V}$ and $w \in W \subseteq Z$ implies $Q_{w,V}$. Hence, we have $(R;(P \cap Q))_{x,V}$. □

As a corollary of this result we obtain the connection to Aumann contact relations that we have formulated as Theorem 15 in Section 9.

Proof (of Theorem 15). Let a multirelation $R : A \leftrightarrow 2^B$ be given. To prove the first claim, assume that R satisfies (K_4'), that is, for all $x \in A$ and $Y, Z \in 2^B$,

$$R_{x,Y} \wedge R_{x,Z} \Leftrightarrow R_{x,Y\cap Z}.$$

We want to show that R is $\cap$-distributive by Theorem 16. To this end, let $x \in A$ and $Y, Z \in 2^B$ such that $R_{x,Y}$ and $R_{x,Z}$. Then we get $R_{x,Y\cap Z}$ by (K_4'). Hence $W := Y \cap Z$ satisfies the condition (1) of Theorem 16 and we are done.

To prove the second claim, we additionally assume that R is up-closed. It remains to show that if R is $\cap$-distributive, then (K_4') follows. The backward implication of (K_4') holds since R is up-closed. For a proof of the forward implication of (K_4'), let $x \in A$ and $Y, Z \in 2^B$ such that $R_{x,Y}$ and $R_{x,Z}$. By Theorem 16 there exists a set $W \in 2^B$ such that $R_{x,W}$ and $W \subseteq Y \cap Z$. Then we also have $R_{x,Y\cap Z}$ since R is up-closed. $\square$

The characterisation of $\cup$-distributive multirelations shown in the following result is not obviously dual to the one for $\cap$-distributive multirelations.

Theorem 17. *A multirelation $R : A \leftrightarrow 2^B$ is $\cup$-distributive if and only if*

$$\forall x \in A : \forall Y, Z \in 2^B : R_{x,Y} \wedge Z \subseteq Y \Rightarrow \exists W \in 2^B : R_{x,W} \wedge (W \subseteq Z \vee W \subseteq Y \setminus Z) \tag{2}$$

Proof. To show the forward implication, let R be $\cup$-distributive. Furthermore, let $x \in A$ and $Y, Z \in 2^B$ such that $R_{x,Y}$ and $Z \subseteq Y$. We define the multirelations $P, Q : B \leftrightarrow 2^B$ by $P_{w,V}$ if and only if $w \in V \cap Z$ and $Q_{w,V}$ if and only if $w \in V \cap (Y \setminus Z)$, for all $w \in B$ and $V \in 2^B$. Then (2) is shown by the following calculation:

$$\begin{array}{rcl}
R_{x,Y} & \Longrightarrow & \exists W \in 2^B : R_{x,W} \wedge W \subseteq Y \\
& \Longleftrightarrow & \exists W \in 2^B : R_{x,W} \wedge \forall w \in W : w \in Y \\
& \Longleftrightarrow & \exists W \in 2^B : R_{x,W} \wedge \forall w \in W : w \in Z \vee w \in Y \setminus Z \\
& \Longleftrightarrow & \exists W \in 2^B : R_{x,W} \wedge \forall w \in W : w \in Y \cap Z \vee w \in Y \cap (Y \setminus Z) \\
& \Longleftrightarrow & \exists W \in 2^B : R_{x,W} \wedge \forall w \in W : P_{w,Y} \vee Q_{w,Y} \\
& \Longleftrightarrow & \exists W \in 2^B : R_{x,W} \wedge \forall w \in W : (P \cup Q)_{w,Y} \\
& \Longleftrightarrow & (R;(P \cup Q))_{x,Y} \\
& \Longleftrightarrow & (R;P \cup R;Q)_{x,Y} \\
& \Longleftrightarrow & (R;P)_{x,Y} \vee (R;Q)_{x,Y} \\
& \Longleftrightarrow & (\exists W \in 2^B : R_{x,W} \wedge \forall w \in W : P_{w,Y}) \vee (\exists W \in 2^B : R_{x,W} \wedge \forall w \in W : Q_{w,Y}) \\
& \Longleftrightarrow & \exists W \in 2^B : R_{x,W} \wedge ((\forall w \in W : P_{w,Y}) \vee (\forall w \in W : Q_{w,Y})) \\
& \Longleftrightarrow & \exists W \in 2^B : R_{x,W} \wedge ((\forall w \in W : w \in Y \cap Z) \vee (\forall w \in W : w \in Y \cap (Y \setminus Z))) \\
& \Longleftrightarrow & \exists W \in 2^B : R_{x,W} \wedge ((\forall w \in W : w \in Z) \vee (\forall w \in W : w \in Y \setminus Z)) \\
& \Longleftrightarrow & \exists W \in 2^B : R_{x,W} \wedge (W \subseteq Z \vee W \subseteq Y \setminus Z)
\end{array}$$

To show the backward implication, assume (2). Let again $P, Q : B \leftrightarrow 2^C$ be multirelations and $x \in A$ and $V \in 2^C$ such that $(R;(P \cup Q))_{x,V}$. Then there exists a set $Y \in 2^B$ such that $R_{x,Y}$ and $(P \cup Q)_{y,V}$, for all $y \in Y$. We define the subset $Z := \{y \in Y \mid P_{y,V}\}$ of Y and get $Y \setminus Z \subseteq \{y \in Y \mid Q_{y,V}\}$. By (2), there exists a set $W \in 2^B$ such that $R_{x,W}$ and $W \subseteq Z$ or $W \subseteq Y \setminus Z$. First, if $W \subseteq Z$, then $(R;P)_{x,V}$ holds, since $P_{w,V}$ for each $w \in W \subseteq Z$. Second, if $W \subseteq Y \setminus Z$, then $(R;Q)_{x,V}$ holds, since $Q_{w,V}$ for each $w \in W \subseteq Y \setminus Z$. In either case we get the desired result $(R;P \cup R;Q)_{x,V}$. $\square$

Again the connection to Aumann contact relations that we have formulated as Theorem 14 in Section 9 is obtained as a corollary of this result.

Proof (of Theorem 14). Let $R : A \leftrightarrow 2^B$ be a multirelation. To prove the first claim, assume that R satisfies (K_4), that is, for all $x \in A$ and $Y, Z \in 2^B$,

$$R_{x,Y\cup Z} \Leftrightarrow R_{x,Y} \vee R_{x,Z}.$$

To show that R is $\cup$-distributive by Theorem 17, let $x \in A$ and $Y, Z \in 2^B$ such that $R_{x,Y}$ and $Z \subseteq Y$. Then $Y = Z \cup (Y \setminus Z)$, whence we have $R_{x,Z\cup(Y\setminus Z)}$. We obtain $R_{x,Z}$ or $R_{x,Y\setminus Z}$ by (K_4). Hence the condition (2) of Theorem 17 is satisfied by $W := Z$ or $W := Y \setminus Z$, respectively.

For the proof of the backward implication of the second claim we assume that R is up-closed and $\cup$-distributive. The backward implication of (K_4) holds since R is up-closed. To prove the forward implication of (K_4), let $x \in A$ and $Y, Z \in 2^B$ such that $R_{x,Y\cup Z}$. Since $Y \subseteq Y \cup Z$, by Theorem 17 there exists a set $W \in 2^B$ such that $R_{x,W}$ and $W \subseteq Y$ or $W \subseteq (Y \cup Z) \setminus Y = Z \setminus Y \subseteq Z$. Then we get $R_{x,Y}$ or $R_{x,Z}$, respectively, since R is up-closed. □

A more compact characterisation can be given under the following finiteness assumption. For a multirelation $R : A \leftrightarrow 2^B$ define the predicate

$$F(R) \iff (\forall x \in A : \forall Y \in 2^B : R_{x,Y} \Rightarrow R_{x,\emptyset} \text{ or } Y \text{ is finite})$$

This predicate is preserved under union, intersection and, as the following result shows, also composition of multirelations.

Theorem 18. *Let $Q : A \leftrightarrow 2^B$ and $R : B \leftrightarrow 2^C$ be multirelations such that $F(R)$. Then we have $F(Q\,;R)$.*

Proof. Let $x \in A$ and $Y \in 2^C$ such that $(Q\,;R)_{x,Y}$. Then there exists a set $Z \in 2^B$ such that $Q_{x,Z}$ and $R_{z,Y}$, for each $z \in Z$. If Y is finite, the claim follows trivially. If Y is infinite, we obtain $R_{z,\emptyset}$, for each $z \in Z$, by the assumption. But then $(Q\,;R)_{x,\emptyset}$ follows. □

The above predicate facilitates a new characterisation of $\cup$-distributive multirelations where, compared with the characterisation in Theorem 17, the number of bound variables is reduced from 4 to 3.

Theorem 19. *A multirelation $R : A \leftrightarrow 2^B$ such that $F(R)$ holds is $\cup$-distributive if and only if*

$$\forall x \in A : \forall Y \in 2^B : R_{x,Y} \Rightarrow R_{x,\emptyset} \vee \exists y \in Y : R_{x,\{y\}} \tag{3}$$

Proof. To show the forward implication, let R be $\cup$-distributive. Furthermore, let $x \in A$ and $Y \in 2^B$ such that $R_{x,Y}$. If $R_{x,\emptyset}$, the claim follows trivially. Otherwise, we get $Y \neq \emptyset$ and that Y is finite, since $F(R)$ holds. Let $y \in Y$, whence $\{y\} \subseteq Y$. By Theorem 17, there exists a set $W \in 2^B$ such that $R_{x,W}$ and $W \subseteq \{y\}$ or $W \subseteq Y \setminus \{y\}$. If $W \subseteq \{y\}$, then $W = \{y\}$, since $W = \emptyset$ would imply $R_{x,\emptyset}$, and we are done. Otherwise we have $R_{x,W}$, where W is strictly smaller than Y. We repeat this process finitely many times until a singleton set is reached.

To show the backward implication, assume (3). By Theorem 17 it suffices to show its condition (2). Let $x \in A$ and $Y, Z \in 2^B$ such that $R_{x,Y}$ and $Z \subseteq Y$. By (3) we obtain $R_{x,\emptyset}$ or $R_{x,\{y\}}$ for some $y \in Y$. If $R_{x,\emptyset}$ we are done since $\emptyset \subseteq Z$. If $R_{x,\{y\}}$ for some $y \in Y$, then we have either $y \in Z$, in which case $\{y\} \subseteq Z$ and we are done, or we have $y \notin Z$, in which case $y \in Y \setminus Z$, whence $\{y\} \subseteq Y \setminus Z$ and we are also done. □

In the next theorem, the finiteness assumption allows us also to show that $\cup$-distributivity is preserved under composition of multirelations.

Theorem 20. *Let $Q : A \leftrightarrow 2^B$ and $R : B \leftrightarrow 2^C$ be $\cup$-distributive multirelations such that $F(Q)$ and $F(R)$ hold. Then $Q\,;R$ is $\cup$-distributive.*

Proof. By Theorem 18 we obtain $F(Q\,;R)$. Hence, by Theorem 19 it suffices to show its condition (3) for $Q\,;R$. To that end, let $x \in A$ and $Y \in 2^C$ such that $(Q\,;R)_{x,Y}$. Then there exists a set $Z \in 2^B$ such that $Q_{x,Z}$ and $R_{z,Y}$ for each $z \in Z$. By Theorem 19 we obtain $Q_{x,\emptyset}$ or $Q_{x,\{z\}}$, for some $z \in Z$. If $Q_{x,\emptyset}$, then we have $(Q\,;R)_{x,\emptyset}$ and we are done. Otherwise, we get $Q_{x,\{z\}}$ and $R_{z,Y}$. Again by Theorem 19 we obtain $R_{z,\emptyset}$ or $R_{z,\{y\}}$, for some $y \in Y$. If $R_{z,\emptyset}$, we get $(Q\,;R)_{x,\emptyset}$ and we are done. Otherwise, we get $R_{z,\{y\}}$, whence $(Q\,;R)_{x,\{y\}}$ and we are also done. □

We end this section with a counterexample showing that $\cup$-distributivity is not preserved under composition in general.

Theorem 21. *There are $\cup$-distributive multirelations Q and R such that $Q\,;R$ is not $\cup$-distributive.*

Proof. We consider the state space $\mathbb{N}$ and define multirelations $P, Q, R : \mathbb{N} \leftrightarrow 2^{\mathbb{N}}$ as follows:

$$
\begin{aligned}
P &:= \{(n, \mathbb{N}) \mid n \in \mathbb{N}\} \\
Q &:= \{(n, S) \mid n \in \mathbb{N} \wedge S \in 2^{\mathbb{N}} \wedge S \text{ is infinite}\} \\
R &:= \{(n, \{n\}) \mid n \in \mathbb{N}\} \cup P
\end{aligned}
$$

To see that Q is $\cup$-distributive, we apply Theorem 17. Let $n \in \mathbb{N}$ and $Y, Z \in 2^{\mathbb{N}}$ be given such that $Q_{n,Y}$ and $Z \subseteq Y$. Then Y is infinite. If Z is infinite, then this yields $Q_{n,Z}$ and we are done. If Z is finite, then $Y \setminus Z$ is a infinite set, whence we obtain $Q_{n,Y\setminus Z}$ and we are also done.

To see that R is $\cup$-distributive, we again apply Theorem 17. Let $n \in \mathbb{N}$ and $Y, Z \in 2^{\mathbb{N}}$ be given such that $R_{n,Y}$ and $Z \subseteq Y$. Then we get $n \in Y$ and $R_{n,\{n\}}$. If $n \in Z$, then we have $\{n\} \subseteq Z$ and we are done. Otherwise, we get $n \in Y \setminus Z$, whence $\{n\} \subseteq Y \setminus Z$ and we are done, too.

We next show that $P = Q\,;R$ holds. Clearly, $P \subseteq Q\,;R$ since $Q_{n,\mathbb{N}}$ and $R_{n,\mathbb{N}}$, for each $n \in \mathbb{N}$. For the converse inclusion, assume $(Q\,;R)_{n,T}$. Then there exists a set S such that $Q_{n,S}$ and $R_{m,T}$, for each $m \in S$. Hence S is infinite and we obtain $T = \{m\}$ or $T = \mathbb{N}$, for each $m \in S$. This can only be the case if $T = \mathbb{N}$. Thus, we have $P_{n,T}$.

Finally, we show that P is not $\cup$-distributive by using Theorem 17. Define $n := 1$ and $Y := \mathbb{N}$ and $Z := \{1\}$. This implies $P_{n,Y}$ and $Z \subseteq Y$. Let $W \in 2^{\mathbb{N}}$ such that $P_{n,W}$. Then we have $W = \mathbb{N}$ and neither $\mathbb{N} \subseteq Z$ nor $\mathbb{N} \subseteq Y \setminus Z$. $\square$

11. Conclusion

In this paper we have investigated multirelations using relation algebras and more general algebraic structures. In particular, we have considered various properties of multirelations that have been used in applications and we have studied transitive closures, closure properties and Aumann contacts.

Unlike the composition operations of Parikh and Peleg, which are not associative for arbitrary multirelations, the composition operation studied in Section 4 is associative but fails to be $\subseteq$-isotone in its second argument. It therefore seems challenging to describe these three composition operations in a uniform way.

The complement operation of multirelations has not received much attention so far, likely because much previous research has focused on up-closed multirelations, which are not closed under complement. It has proved to be useful in describing properties such as down-closed multirelations, and it remains to be seen if additional axioms about the interaction of complement and composition are needed to derive inequalities such as $\overline{x\overline{1}}\,\overline{1} \leq \overline{x}$ which hold for arbitrary multirelations. Down-closed multirelations are also important in connection with Peleg's composition operation; this will be discussed in further work.

We finally observe that the connection between up-closed and down-closed multirelations is of a different kind than the duality between some of the other properties, say, tests and co-tests. The former is obtained by substituting the complement of a relation in the equation defining the property, while the latter is obtained by applying the dual operation to the whole expression on both sides of the equation.

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